



Optimization-based computation of analytic interpolants of bounded complexity[☆]

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Abstract

This paper provides a unifying algorithm for computing any analytic interpolant of bounded complexity. Such computation can be performed by solving an optimization problem, due to a theorem by Georgiou and Lindquist. This optimization problem is numerically solvable by a continuation method. The proposed numerical algorithm is useful, among other cases, for designing a low-degree controller for a benchmark problem in robust control. The algorithm unifies previously developed algorithms for the Carathéodory extension and the Nevanlinna–Pick interpolation to one for more general interpolation problems.

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1. Introduction

It has been recognized for decades that classical analytic interpolation theory, such as Carathéodory extension and Nevanlinna–Pick interpolation [26,2,17], has interesting applications in systems and control. Although the classical theory provides the characterization of all interpolants, it is of little use in characterizing all interpolants of a certain degree, or more

generally, complexity. The complexity of interpolants is of major concern in applications, since it relates to the complexity of various components/devises. Therefore, it is important to develop the theory and computational algorithms for analytic interpolation with complexity constraint.

The first work on analytic interpolation with degree constraint was done by Georgiou in [18–20]. He showed the existence of an interpolant of bounded degree for each choice of spectral zeros (or dissipative polynomial). He also conjectured that each choice of spectral zeros uniquely determines such an interpolant. This conjecture was proven to be true, together with a much stronger assertion on the smoothness of the

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parameterization, in [13], which revitalized the research field. Later constructive proofs based on convex optimization were given in [9,7]. In the recent paper [23], the optimization approach in [9,7] was extended to a generalized analytic interpolation problem with complexity constraint, including both the Carathéodory extension and the Nevanlinna–Pick interpolation as special cases.

To utilize the theory in [9,7,23], robust numerical algorithms to solve the optimization problem are indispensable. In [16,24,5], we have developed such algorithms for the theory of [9,7], by means of a continuation method. In this paper, following the extension of the theory in [9,7] to the one in [23], we will show that the numerical technique in [16,24,5] still applies to the general theory in [23]. This is the first contribution of this paper.

The importance of the theory in [23] and the algorithm proposed in this paper can be substantiated by engineering applications. Such applications have been presented in [3] for spectral estimation using orthonormal basis functions and in [6] for pre-filtered AR-estimation. In this paper, we provide an application to sensitivity shaping in robust control. Using an example, we will show that the controller degree becomes much lower than with a conventional H^∞ control method thanks to the theory in [23], and that such low degree controllers can be computed by the proposed algorithm which can deal with not only interpolation but also derivative interpolation conditions. This is the second contributions of this paper.

The outline of this paper is as follows. In Section 2, the problem of analytic interpolation with complexity constraint is formulated and the theory in [23] for this problem is reviewed. Based on the theory, Section 3 formulates a new, more tractable optimization problem, equivalent to the original problem in a certain sense, and gives an algorithm for solving it. Section 4 provides an example of sensitivity shaping in robust control and shows how the proposed computational algorithm can be used.

2. Analytic interpolation with complexity constraint

In this section, we formulate a problem of finding spectral densities subjected to both moment and

complexity conditions. This problem reduces to several analytic interpolation problems in special cases. We also state a theorem from [23] for this problem concerning the existence and the uniqueness of the solution.

First we will introduce some notation. Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times 1}$ be given such that A has all eigenvalues in the open unit disc and (A, B) is a reachable pair. The pair (A, B) constitutes a transfer function

$$G(z) := (I - zA)^{-1}B. \tag{1}$$

Let $G^*(z) := \overline{G(\bar{z}^{-1})}^T$ denote the point-wise complex conjugate transpose. Using the given G in (1), define the following set of Hermitian matrices:

$$\mathfrak{L}_+ := \left\{ A \in \mathbb{C}^{n \times n} : \begin{array}{l} A = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{i\theta})\Phi(e^{i\theta}) \\ \times G^*(e^{i\theta}) d\theta, \\ \Phi = \Phi^* \in \mathcal{C}(\mathbb{T}) \\ G^*AG \in \mathcal{C}_+(\mathbb{T}) \end{array} \right\},$$

where $\mathcal{C}(\mathbb{T})$ is the space of continuous real-valued functions on the unit circle \mathbb{T} and $\mathcal{C}_+(\mathbb{T})$ its subset of positive functions. Also, define the set

$$\mathfrak{S}_+ := \left\{ \Sigma \in \mathbb{C}^{n \times n} : \Sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{i\theta})\Phi(e^{i\theta}) \times G^*(e^{i\theta}) d\theta, \Phi = \Phi^* \in \mathcal{C}_+(\mathbb{T}) \right\}. \tag{2}$$

Due to [22, Theorem 1], \mathfrak{S}_+ can equivalently be written as

$$\mathfrak{S}_+ = \{ \Sigma \in \mathbb{C}^{n \times n} : 2\Sigma = WE + EW^* > 0, W \in \mathfrak{B}(A) \},$$

where E is the reachability Grammian of (A, B) and $\mathfrak{B}(A) := \{W \in \mathbb{C}^{n \times n} : AW = WA\}$ is the set of matrices which are polynomials in A and thus a n dimensional linear space over the field of complex numbers. The matrix Σ is called a generalized *Pick matrix*. Clearly $\mathfrak{S}_+ \subset \mathfrak{L}_+$ since the positive definiteness of Σ implies $G^*(e^{i\theta})\Sigma G(e^{i\theta}) > 0$ for all θ , and hence $G^*\Sigma G \in \mathcal{C}_+(\mathbb{T})$.

The problem of interest can now be stated as follows.

Problem 2.1. *Let G and $\Sigma \in \mathfrak{S}_+$ (or equivalently the triple (A, B, W)) be given as above. For each*

$\Psi = \Psi^* \in \mathcal{C}_+(\mathbb{T})$, find any $\Phi \in \mathcal{C}_+(\mathbb{T})$ of the form

$$\Phi = \frac{\Psi}{G^* \Lambda G}, \quad \Lambda \in \mathfrak{Q}_+, \quad (3)$$

which satisfies

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{i\theta}) \Phi(e^{i\theta}) G^*(e^{i\theta}) d\theta = \Sigma. \quad (4)$$

Although the spectral density Φ is neither analytic nor interpolates, the problem can be interpreted as an *analytic interpolation problem* for the following reason. Due to Riesz–Herglotz representation theorem [2, p. 91], the density function Φ corresponds to the Carathéodory function

$$F(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \Phi(e^{i\theta}) d\theta,$$

which is *analytic* in the open unit disc and maps the open unit disc into the open right half-plane. The density is then the real part of the analytic function: $\Phi(z) = 2\text{Re}\{F(z)\}$. Condition (4) can be stated in terms of the analytic function F as the *interpolation condition*; see [22]. For instance, taking

$$A = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & \ddots & \ddots & & & \\ & & & 1 & 0 & \\ & & & & & \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$W = \begin{bmatrix} w_0 & & & & & \\ w_1 & w_0 & & & & \\ \vdots & \ddots & \ddots & & & \\ w_{n-1} & \cdots & w_1 & w_0 & & \end{bmatrix}, \quad (5)$$

corresponds to the conditions $F(0)/k! = w_k$, $k = 0, \dots, n - 1$, in the Carathéodory extension problem, and

$$A = \begin{bmatrix} p_1 & & & & \\ & \ddots & & & \\ & & & p_n & \\ & & & & \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$W = \begin{bmatrix} w_1 & & & & \\ & \ddots & & & \\ & & & & w_n \end{bmatrix}, \quad (6)$$

to the conditions $F(p_k) = w_k$, $k = 1, \dots, n$, in the Nevanlinna–Pick problem. Also derivative conditions on F at arbitrary points in the unit disc (see, e.g., [15])

can also be expressed as (4) with appropriate A, B and W ; see [23]. In fact, the interpolation conditions can equivalently be written in the operator theory style as $F(A) = W$.

Constraint (3) on the density Φ can be interpreted as a *complexity constraint*. In the case when Ψ is a rational function, it degenerates to a *degree constraint* of F , studied in, e.g., [13,9,7].

In [23] Georgiou and Lindquist show the existence and the uniqueness of a solution to Problem 2.1 by considering an approximation problem. Furthermore, they provide a constructive way to determine the unique solution.

Theorem 2.2 (Georgiou [23]). *Let G and $\Sigma \in \mathfrak{S}_+$ (or equivalently the triple (A, B, W)) be given as above. For each $\Psi = \Psi^* \in \mathcal{C}_+(\mathbb{T})$, there exists a unique $\Phi = \Phi^* \in \mathcal{C}_+(\mathbb{T})$ which minimizes*

$$\mathbb{S}(\Psi \parallel \Phi) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \log \frac{\Psi(e^{i\theta})}{\Phi(e^{i\theta})} d\theta, \quad (7)$$

subject to (4). The minimizer takes the form (3) where Λ is the unique minimizer of the dual functional

$$\mathbb{J}_\Psi(\Lambda) := \text{trace}(\Lambda \Sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \times \log(G^*(e^{i\theta}) \Lambda G(e^{i\theta})) d\theta, \quad (8)$$

over $\Lambda \in \mathfrak{Q}_+$.

Here $\mathbb{S}(\Psi \parallel \Phi)$ is the *Kullback–Leibler divergence* between spectral densities. It can be viewed as a generalization of entropy (set $\Psi = 1$). We emphasize that the global strict convexity of the dual functional (8) also gives the uniqueness of the solution to Problem 2.1. Next, we will give a numerical algorithm for solving the dual problem.

3. An algorithm for solving the optimization problem

This section proposes a numerical algorithm to solve the optimization problem

$$(\text{OPT}_1) \quad \min_{\Lambda \in \mathfrak{Q}_+} \mathbb{J}_\Psi(\Lambda),$$

where the functional \mathbb{J}_Ψ is defined in (8). Since the domain \mathfrak{Q}_+ is a convex set and the cost functional \mathbb{J}_Ψ

is a convex function, (OPT₁) is a convex optimization problem. In fact, it has a unique *interior point* minimizer. However, the problem may not, in some cases, be accurately solvable. This can be attributed to the unbounded gradient of the functional at the boundary of the feasible region. This property is numerically undesirable since it leads to ill-conditioning of systems of linear equations arising in Newton iterations; see discussions in [16,24].

To avoid such undesirable properties we, in Section 3.1, transform the optimization problem (OPT₁) into a new optimization problem. This new problem is a generalization of the ones in [16,24,5], but of the same type. To be more precise, it is in general non-convex, but has better numerical properties. Thus, to solve the new problem, we utilize the same numerical algorithm based on a continuation method as in [16,24,5]. The numerical properties and the algorithm will be sketched in Section 3.2.

3.1. An equivalent optimization problem

Here we will show that the optimization problem (OPT₁) is equivalent (in the sense explained in Proposition 3.1) to the optimization problem

$$(OPT_2) \quad \min_{\alpha \in \mathfrak{Q}_+} J_\Psi(\alpha),$$

where the domain is the Schur stability region defined by

$$\mathfrak{Q}_+ := \left\{ \begin{array}{l} \alpha = [\alpha_0 \ \alpha_1 \ \dots \ \alpha_{n-1}]^T \\ \in \mathbb{R} \times \mathbb{C}^{n-1} : \alpha_0 > 0 \\ \alpha(z) := \alpha_0 \\ \quad + \alpha_1 z + \dots + \alpha_{n-1} z^{n-1} \neq 0, \forall z \in \overline{\mathbb{D}} \end{array} \right\}$$

with the closed unit disc $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. The new cost functional is defined as

$$J_\Psi(\alpha) := \alpha^* K \alpha - 2 \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \log |\alpha(e^{i\theta})| d\theta,$$

where the $n \times n$ positive definite Hermitian matrix K is defined by

$$K := L^{-*} \bar{\Gamma}^{-1} \Sigma^T \Gamma^{-T} L^{-1}. \tag{9}$$

Here Γ is the reachability matrix of a reachable pair (A, B) and L is the lower triangular Toeplitz matrix

defined as

$$L := \begin{bmatrix} 1 & 0 & \dots & 0 \\ \tau_1 & 1 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tau_{n-1} & \dots & \tau_1 & 1 \end{bmatrix}$$

with $\tau(z) := 1 + \tau_1 z + \dots + \tau_n z^n := \det(I - zA)$.

For each $\Lambda \in \mathfrak{Q}_+$, since $G^* \Lambda G \in \mathcal{C}_+(\mathbb{T})$ and by uniqueness up to sign of spectral factorization, there is a unique $\alpha \in \mathfrak{Q}_+$ such that

$$G^*(z) \Lambda G(z) = \frac{\alpha(z) \alpha^*(z)}{\tau(z) \tau^*(z)}. \tag{10}$$

Note that without the conditions $\alpha_0 \in \mathbb{R}$ and $\alpha_0 > 0$, α in (10) is not unique since $\alpha(z) \alpha^*(z) = (e^{i\theta} \alpha(z)) (e^{i\theta} \alpha(z))^*$ for all $\theta \in \mathbb{R}$. Let $H : \mathfrak{Q}_+ \rightarrow \mathfrak{Q}_+$ denote the corresponding map. The optimization problems (OPT₁) and (OPT₂) are equivalent in the sense of the following proposition.

Proposition 3.1. *The matrix $\hat{\Lambda}$ is the unique solution to (OPT₁) if and only if $\hat{\alpha}$ is the unique solution to (OPT₂) where $\hat{\alpha} = H(\hat{\Lambda})$.*

Proposition 3.1 tells us that we might as well solve (OPT₂). The main advantage of solving (OPT₂) rather than (OPT₁) is better numerical properties, see [16,24]. Also, in the *self-conjugate case*, we get a real parameterization and avoid complex arithmetics. The self-conjugate case is when it is possible to restate the problem in real A, B and W . For instance, in Nevanlinna–Pick interpolation, it corresponds to having the constraints $F(\bar{z}_k) = \bar{w}_k$ and $F(z_k) = w_k$ simultaneously.

In order to prove the proposition, we will need the following lemma.

Lemma 3.2. *The map H is a bijection. Moreover, for each $\Lambda \in \mathfrak{Q}_+$,*

$$\text{trace}(\Lambda \Sigma) = \alpha^* K \alpha,$$

where $\alpha = H(\Lambda)$ and K is defined in (9).

Proof of Lemma 3.2. It can be shown that \mathfrak{Q}_+ has real dimension $2n - 1$ [23, Lemma 4], and clearly the

same holds for \mathfrak{A}_+ and

$$\mathfrak{Q}_+ := \{q \in \mathbb{C}^{n \times 1} : \operatorname{Re}\{q^T G(z)\} > 0, \\ \forall z \in \mathbb{T}, \operatorname{Im}\{q^T B\} = 0\}.$$

Hence they are all open sets in \mathbb{R}^{2n-1} . Since \mathfrak{Q}_+ and \mathfrak{A}_+ are also convex, they are Euclidean (diffeomorphic to \mathbb{R}^{2n-1}) (see [11, Lemma 6.7] for a rigorous proof). It can also be shown that \mathfrak{A}_+ is Euclidean [10, p. 2306].

Now, $H = H_2 \circ H_1$, where $H_1 : \mathfrak{Q}_+ \rightarrow \mathfrak{A}_+$ is the finite-dimensional map sending A to q via $q^T G + (q^T G)^* = G^* A G$ and $H_2 : \mathfrak{A}_+ \rightarrow \mathfrak{Q}_+$ maps q to α via spectral factorization. By Georgiou [23, Lemma 8], the map H_1 is injective. Now, for all compact $K \subset \mathfrak{Q}_+$ one can show that the inverse images $H_1^{-1}(K)$ are bounded by a contradiction argument. Taking any sequence $A_k \in H_1^{-1}(K)$ converging to a $A_0 \in \overline{\mathfrak{Q}_+}$, $A_0 \in H_1^{-1}(K)$ unless it lies on the boundary of \mathfrak{Q}_+ since H_1 is continuous and injective. However, since H_1 maps the boundary of \mathfrak{Q}_+ into the boundary of \mathfrak{A}_+ and K is compact, A_0 cannot be on the boundary. Therefore $H_1^{-1}(K)$ is also closed and hence compact. Therefore H_1 is proper. Consequently, since H_1 maps between two Euclidean spaces of the same dimension, it is a homeomorphism [12, Lemma 2.3] and hence a bijection. Clearly H_2 is also bijective, and hence so is H .

Finally, there is a $C \in \mathbb{C}^n$ such that

$$\frac{\alpha(z)}{\tau(z)} = C^T G(z),$$

where the coefficients are related by the linear relation $C = \Gamma^{-T} L^{-1} \alpha$. Therefore, proceeding along the lines of [23, p. 2915], we have

$$\begin{aligned} & \operatorname{trace}(A \Sigma) \\ &= \operatorname{trace} \left(A \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{i\theta}) \Phi(e^{i\theta}) G^*(e^{i\theta}) d\theta \right), \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{trace}(A G(e^{i\theta}) \Phi(e^{i\theta}) G^*(e^{i\theta})) d\theta, \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G^*(e^{i\theta}) A G(e^{i\theta}) \Phi(e^{i\theta}) d\theta, \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} C^T G(e^{i\theta}) G^*(e^{i\theta}) \overline{C} \Phi(e^{i\theta}) d\theta, \\ &= C^T \overline{C} \Sigma = C^* \Sigma^T C = \alpha^* K \alpha. \end{aligned}$$

This proves the second part of the lemma. \square

Now the proof of Proposition 3.1 is straightforward.

Proof of Proposition 3.1. Let $A \in \mathfrak{Q}_+$ and $\alpha = H(A)$. Using factorization (10) and Lemma 3.2, we have

$$\begin{aligned} \mathbb{J}_\Psi(A) &= \operatorname{trace}(A \Sigma) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \\ &\quad \times \log(G^*(e^{i\theta}) A G(e^{i\theta})) d\theta, \\ &= \alpha^* K \alpha - \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) (\log(\alpha(e^{i\theta}) \alpha^*(e^{i\theta})) \\ &\quad - \log(\tau(e^{i\theta}) \tau^*(e^{i\theta}))) d\theta, \\ &= \alpha^* K \alpha - 2 \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi(e^{i\theta}) \\ &\quad \times \log |\alpha(e^{i\theta})| d\theta + c, \\ &= J_\Psi(\alpha) + c, \end{aligned}$$

where $c := -\int_{-\pi}^{\pi} \Psi(e^{i\theta}) \log(\tau(e^{i\theta}) \tau^*(e^{i\theta})) d\theta / 2\pi$.

Let \hat{A} be the unique optimizer of (OPT₁) in \mathfrak{Q}_+ :

$$\mathbb{J}_\Psi(\hat{A}) < \mathbb{J}_\Psi(A), \quad \forall A \in \mathfrak{Q}_+, \quad A \neq \hat{A}.$$

Then with $\hat{\alpha} = H(\hat{A})$, using the bijectivity in Lemma 3.2, we equivalently have

$$\begin{aligned} J_\Psi(\hat{\alpha}) + c &= \mathbb{J}_\Psi(\hat{A}) < \mathbb{J}_\Psi(A) = J_\Psi(\alpha) + c, \\ \forall \alpha \in \mathfrak{A}_+, \quad \alpha &\neq \hat{\alpha}. \end{aligned}$$

This concludes the proof. \square

3.2. An algorithm for solving the new optimization problem

The optimization problem (OPT₂) is a generalization of the optimization problems that appeared in [16,24,5] in two ways. Firstly, interpolation conditions are given in the more general formulation using the reachable pair (A, B) , and secondly, the class of Ψ . In fact, while Ψ in [16,24,5] was on the form $\Psi = (\rho \rho^*) / (\tau \tau^*)$ for some stable polynomial ρ of degree at most $n - 1$, in this paper, $\Psi = \Psi^*$ is arbitrary in $\mathcal{C}_+(\mathbb{T})$. Still, the optimization problem (OPT₂) is of the same type as in [16,24,5]. It also shares the properties of having a unique global minimizer and being locally convex around the global minimizer. To avoid getting stuck in local, but not global, minima, we will apply the same continuation method as in [16,24,5]. In fact, we will define a *unique* trajectory from an initial point to the global optimum. By iterating close

enough to the trajectory, we will avoid getting stuck in a local minimum which is not global. In this section we summarize the algorithm.

Consider a family of optimization problems

$$\{\min_{\alpha \in \mathfrak{A}_+} J_{\Psi_\lambda}(\alpha) : \Psi_\lambda := 1 + \lambda(\Psi - 1) \in \mathcal{C}_+(\mathbb{T})\},$$

$$\lambda \in [0, 1], \quad (11)$$

which all are of the same type as (OPT₂). For $\lambda = 0$, the optimization problem

$$\min_{\alpha \in \mathfrak{A}_+} J_1(\alpha)$$

has a minimizer in a closed form, called a *maximum entropy solution* [23]. When $\lambda = 1$, $\Psi_1 = \Psi$ and thus the optimization problem

$$\min_{\alpha \in \mathfrak{A}_+} J_\Psi(\alpha)$$

coincides with (OPT₂). For $\lambda \in [0, 1]$, all the solutions $\hat{\alpha}_\lambda$ constitutes a trajectory of minimizers to a family of optimization problems (11), that we will be approximately followed using a predictor–corrector method [1]. In summary we present the algorithm below:

A predictor–corrector algorithm

1. Determine the maximum entropy solution (closed form) and set $\lambda_0 = 0$.
2. (Predictor step) Pick a new $\lambda_{k+1} > \lambda_k$. Compute a predictor step in the direction of the trajectory.
3. (Corrector step) Use Newton iterates to return to the trajectory.
4. Iterate the Predictor and Corrector steps until $\lambda_N = 1$.

Expressions for the predictor direction, the gradient and the Hessian all generalizes directly from [24,5], and are hence omitted here. In each Corrector step we solve one optimization problem in (11), which is of the type (OPT₂).

By increasing λ_k slowly enough, the predicted points will lie in the region of convexity, so that the Newton iterates in the Corrector step converges. In the implementation we check a necessary condition for being in the region of local convexity, namely that the Hessian is positive definite. However, increasing λ_k rapidly to one reduces the number of iteration steps. Sensible rules for choosing the sequence $\{\lambda_k\}$ are given in [16,24].

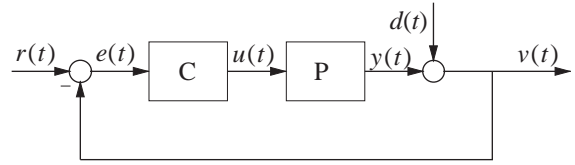


Fig. 1. The feedback system.

4. A low degree controller design for a benchmark problem

In H^∞ controller design, analytic interpolation with complexity constraint has turned out to be particularly useful [7,25,4]. In the conventional approach, one is confined to a particular interpolant and design is performed by insightful choices of weighing functions. By parameterizing a class of H^∞ controllers of a certain degree, weighing functions can be avoided. This gives a new paradigm for tuning H^∞ controllers while keeping controller degrees low.

In this section, we will study the “Flexible Beam” design problem from the textbook *Feedback Control Theory* [14, Sections 10.3 and 12.4]. The study indicates the potential use of the proposed algorithm while showing how the design is performed in a step-by-step manner.

4.1. Formulation of the controller design problem

Consider the standard feedback system depicted in Fig. 1. The plant P is given by

$$P(s) = \frac{-6.4750s^2 + 4.0302s + 175.7700}{s(5s^3 + 3.5682s^2 + 139.5021s + 0.0929)},$$

which has one unstable pole at $s=0$, one non-minimum phase zero at $s = 5.5308$, and one zero at infinity of multiplicity two. The objective is to design a controller C in Fig. 1 which achieves the following conditions:

- (I) C is strictly proper,
- (II) the feedback system is internally stable,
- (III) for a step reference signal $r(t)$,
 - (i) the settling time is less than 8 seconds,
 - (ii) the overshoot is less than 10%, and
 - (iii) the control input fulfills $|u(t)| \leq 0.5$ for all t .

In the book [14], the authors approximate the time domain conditions (III)(i) and (III)(ii) by a frequency domain condition. More concretely, they try to match the frequency response of the sensitivity function $S := (1 + PC)^{-1}$ to an “ideal” sensitivity function given by

$$S_{\text{ideal}}(s) := \frac{s(s + 1.2)}{s^2 + 1.2s + 1},$$

which captures the time domain conditions (III)(i) and (III)(ii). We follow the same strategy, i.e., we state the problem in the frequency domain as designing a sensitivity function S which achieves (I), (II) and

(III') the frequency response of S is similar to that of S_{ideal} .

Here, we do not explicitly include condition (III)(iii). Instead, we need to check the condition after designing an S and, if necessary, redesign an S without an unacceptable degradation of the frequency response of S . Similarly, in the end we need to check conditions (III)(i) and (III)(ii) since they are not guaranteed by (III').

Now, we define a set of sensitivity functions

$$\mathcal{S} := \{S : S \text{ fulfills conditions (I), (II), and } \|S\|_{\infty} < \gamma\}.$$

for some $\gamma > \|S_{\text{ideal}}\|_{\infty}$. Our strategy is to determine an element of \mathcal{S} which solves the control problem.

4.2. Reduction to an analytic interpolation problem with degree constraint

Next, we will show that each element in the set¹

$$\mathcal{P} := \left\{ \Phi = \Phi^* \in \mathcal{C}_+(\mathbb{T}) : \frac{1}{2\pi} \times \int_{-\pi}^{\pi} G(e^{i\theta})\Phi(e^{i\theta})G^*(e^{i\theta})d\theta = \Sigma \right\}, \quad (12)$$

where G is defined in (2), determines a unique element in the set \mathcal{S} . We also determine the matrices A , B and W of Section 2 for our particular control problem. We need the following five steps to do this.

Firstly, conditions (I) and (II) can be expressed using S , see for instance [14], as S being analytic in the

closed right half-plane $\overline{\mathbb{C}_+}$ and satisfying conditions at unstable poles, non-minimum phase zeros and infinite zeros. For our case we get

$$S(0) = 0, \quad S(5.5308) = 1, \quad S(\infty) = 1, \\ \frac{d}{ds} S(s^{-1}) \Big|_{s=0} = \frac{d^2}{ds^2} S(s^{-1}) \Big|_{s=0} = 0.$$

Note that the condition on the highest derivative of S will make the corresponding controller strictly proper. Therefore the set \mathcal{S} becomes

$$\mathcal{S} = \left\{ S : \begin{array}{l} S \text{ is analytic in } \overline{\mathbb{C}_+}, \|S\|_{\infty} < \gamma \\ S(0) = 0, \quad S(5.5308) = 1, \quad S(\infty) = 1 \\ \frac{d}{ds} S(s^{-1}) \Big|_{s=0} = \frac{d^2}{ds^2} S(s^{-1}) \Big|_{s=0} = 0 \end{array} \right\}. \quad (13)$$

In the second step, we introduce the new function $\hat{S}(s) := S(s^{-1})$. Then we can express the interpolation conditions for the function and its derivatives. The set \mathcal{S} has a one-to-one correspondence to the new set

$$\hat{\mathcal{S}} := \left\{ \hat{S} : \begin{array}{l} \hat{S} \text{ is analytic in } \overline{\mathbb{C}_+}, \|\hat{S}\|_{\infty} < \gamma \\ \hat{S}(\infty) = 0, \quad \hat{S}(1/5.5308) = 1, \quad \hat{S}(0) = 1 \\ \frac{d}{ds} \hat{S}(0) = \frac{d^2}{ds^2} \hat{S}(0) = 0 \end{array} \right\}.$$

In the third step we consider the bilinear transformations of the domain and the range as

$$\hat{F}(z) := \frac{\gamma + \hat{S}(1 - z)}{\gamma - \hat{S}(1 + z)}.$$

Transforming the data according to the bilinear transformations, the set $\hat{\mathcal{S}}$ has a one-to-one correspondence to the set

$$\hat{\mathcal{F}} := \left\{ \hat{F} : \begin{array}{l} \hat{F} \text{ is analytic in } \overline{\mathbb{D}}, \hat{F} + \hat{F}^* \in \mathcal{C}_+(\mathbb{T}) \\ \hat{F}(-1) = 1, \quad \hat{F}(0.6938) = 3.5, \\ \hat{F}(1) = 3.5 \\ \frac{d}{ds} \hat{F}(1) = \frac{d^2}{ds^2} \hat{F}(1) = 0 \end{array} \right\}.$$

In the fourth step we define $F(z) := \hat{F}(z/\kappa)$ for some $|\kappa| < 1$. The reason for doing this is that some interpolation points lie on the unit circle. This corresponds to eigenvalues of A in (1) on the unit circle, and this case is not covered with the theory in [23]. The smaller κ the worse approximation, but typically, the better behaved optimization problem. Here, we take $\kappa = 0.90$

¹ The complexity constraint (3) will be imposed on Φ later.

and get the solution set

$$\mathcal{F} := \left\{ F : \begin{array}{l} F \text{ is analytic in } \overline{\mathbb{D}}, F + F^* \in \mathcal{C}_+(\mathbb{T}) \\ F(-0.90) = 1, F(0.6244) = 3.5, \\ F(0.90) = 3.5 \\ \frac{d}{ds} F(0.90) = \frac{d^2}{ds^2} F(0.90) = 0 \end{array} \right\}.$$

Notice that the relation $\{\hat{F} : \hat{F}(z) = F(\kappa z), F \in \mathcal{F}\} \subset \hat{\mathcal{F}}$ holds.

In the last step we define the function $\Phi := F + F^*$. The set \mathcal{F} is then in one-to-one correspondence with the set \mathcal{P} in (12), if we take

$$A = \begin{pmatrix} 0.90 & 0 & 0 & 0 & 0 \\ 1 & 0.90 & 0 & 0 & 0 \\ 0 & 1 & 0.90 & 0 & 0 \\ 0 & 0 & 0 & -0.90 & 0 \\ 0 & 0 & 0 & 0 & 0.63 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad W = \begin{pmatrix} 3.5 & 0 & 0 & 0 & 0 \\ 0 & 3.5 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3.5 \end{pmatrix}.$$

To bound the controller degree, we introduce a degree constraint on S , see [25]. To this end, note that due to Theorem 2.2 and Lemma 3.2, for each given $\Psi = \Psi^* \in \mathcal{C}_+(\mathbb{T})$ there exists a unique element of the form

$$\Phi = \frac{\Psi}{G^*AG} = \frac{\Psi}{\alpha\alpha^*/\tau\tau^*},$$

in the set \mathcal{P} . If we restrict the form of Ψ to $\Psi = (\rho\rho^*)/(\tau\tau^*)$, where ρ is a stable polynomial of degree at most four, then we can guarantee that the McMillan degree of S is at most four. In this case, we get the set of degree constrained spectral densities

$$\hat{\mathcal{P}} := \left\{ \Phi \in \mathcal{P} : \Phi = \frac{\rho\rho^*}{\alpha\alpha^*}, \deg \rho \leq 4, \rho(z) \neq 0, \forall z \in \{z : |z| \geq 1\} \right\}.$$

This corresponds to the solution set to an analytic interpolation problem with degree constraint parameterized by ρ . For each ρ , we can determine the corresponding element Φ in $\hat{\mathcal{P}}$ by using the algorithm of Section 3.2 and successively compute the unique S in \mathcal{S} . The ρ will serve as a tuning parameter in the controller design.

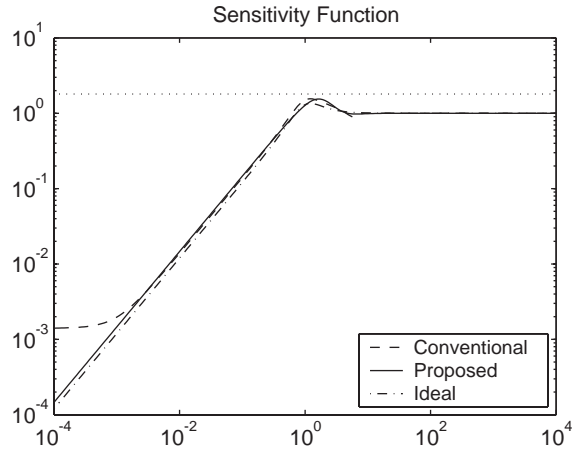


Fig. 2. The frequency responses of sensitivity functions for the conventional design, our design and the ideal function. The dotted line is the upper bound γ of $|S|$.

4.3. Controller design in the frequency domain

Here we tune our design parameters γ and ρ in order to fulfill condition (III'). First, we determine a value of γ larger than $\|S_{\text{ideal}}\|_\infty$ which will guarantee a certain level of robustness. In this problem, we pick $\gamma = 1.8$, see Fig. 2.

Next, we will choose a ρ of degree four to achieve all the specifications. Using the strategies of [25] together with some trial-and-error, we pick the roots of ρ as $z = 0.4373 \pm 0.7866i, 0.6750$, and 0.9000 . With the bilinear transformations of Section 4.2 the corresponding roots in the domain of S are $s = \pm 1.7i, 7$, and ∞ . The complex conjugated pair of roots on the imaginary axis yields a peak of $|S|$ close to that of $|S_{\text{ideal}}|$, see [25]. The other roots are chosen so that the magnitude of the control signal fulfills the condition (III)(iii).² Multiplication of ρ with a constant does not affect S . For this choice of ρ , the function S in \mathcal{S} and the controller C becomes

$$S(s) = \frac{s^4 + 15.24s^3 + 64.42s^2 + 132.58s}{s^4 + 15.24s^3 + 64.42s^2 + 116.21s + 90.49},$$

$$C(s) = \frac{12.63s^3 + 9.016s^2 + 352.5s + 0.2347}{s^4 + 20.15s^3 + 139.2s^2 + 448.8s + 650.7}. \tag{14}$$

²The choice of roots of ρ for condition (III)(iii) is quite heuristic at this point.

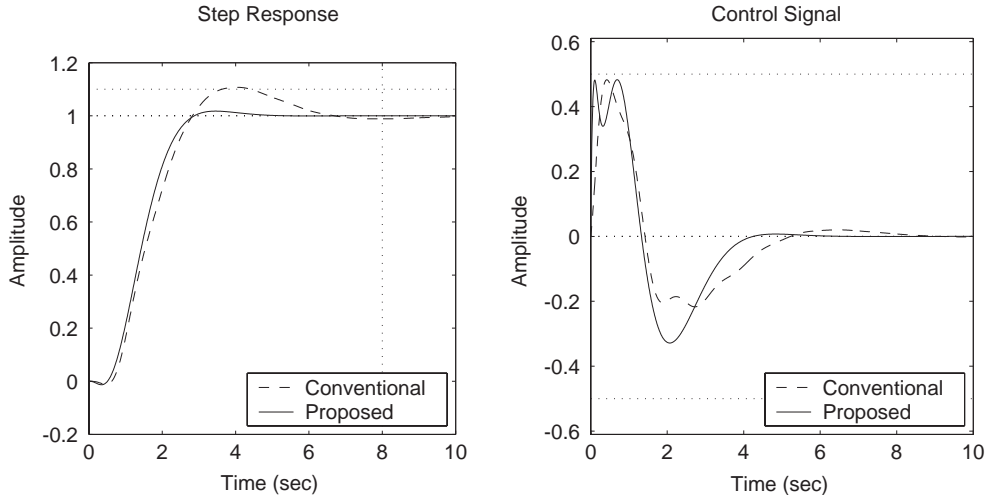


Fig. 3. The step response $y(t)$ for the closed loop systems and the corresponding control signal $u(t)$ for the conventional and the proposed designs.

Table 1

The performance for a conventional H^∞ design and the proposed design.

	Conventional	Proposed
Controller degree	8	4
Rise time (s)	1.55	1.46
Overshoot	1.11	1.02
Settling time (s)	5.41	2.49
Max $ u $	0.48	0.48

In Fig. 2 the frequency responses of the proposed design, the design of the book [14] and of S_{ideal} are plotted. One can see that our sensitivity function almost overlaps S_{ideal} and thus in agreement with (III').

4.4. Performance comparison in the time domain

Now we verify the achievement of the given specification in the time domain, and compare controller (14) with the controller designed in [14, p. 215]. The step responses of the closed-loop systems and the corresponding control signals are plotted in Fig. 3. In Table 1, we can see that our design not only meets the specifications but also provides a better performance than the conventional design. In addition, our controller is of half the degree.

5. Conclusions

In this paper, we have proposed a numerical algorithm for computing any solution to a general class of analytic interpolation problems with complexity constraint. The computation amounts to solving a convex optimization problem which has numerical undesirable properties. We have transformed the problem into an optimization problem which is non-convex but solvable by means of a numerical continuation method. The proposed algorithm has successfully been applied to a benchmark problem in robust control.

At this stage, the algorithm only works for the scalar case. In this context, we note that the existence proof in [18] was also done for the multivariate case. It is important to develop algorithms for multivariate versions of analytic interpolation with complexity constraint. One such algorithm was developed based on the theory on matrix-valued interpolation with degree constraint in [4]. Other extensions to multivariate cases, such as tangential interpolation (see e.g. [15]), will be an important subject of future research. In addition, interpolants with complexity constraint having boundary spectral zeros are important in applications. In [21] it was shown that the parameterization in terms of spectral zeros can be extended to the boundary, and in [8] it was shown in a more general setting that the

optimization approach can also be extended to the boundary. However, a computational algorithm for the theory is still to be developed. Furthermore, it is very important and interesting to study how to tune the H^∞ controllers in this framework. This work is initialized in [25] but more research effort is needed.

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