STABILIZATION OF OBSERVER-BASED ACTUATOR FAULT-TOLERANT CONTROL SYSTEMS WITH UNCERTAINTIES, ACTUATOR SATURATION AND DISTURBANCES: AN LMI APPROACH

Jinhua Fan\(^1,2\) and Zhiqiang Zheng\(^1\)
1. College of Mechatronic Engineering and Automation, National University of Defense Technology, Changsha, Hunan, China
E-mail: fjhcom@gmail.com

Youmin Zhang\(^2\)
2. Dept. of Mechanical & Industrial Engineering, Concordia University, Montreal, Quebec, Canada
ymzhang@encs.concordia.ca

ABSTRACT

An observer-based design method of fault-tolerant controller for uncertain linear systems subject to actuator faults, saturation and bounded disturbances is provided in this paper. A state feedback controller is designed to maximize the attraction domain with the state variables estimated by a Luenberger observer. The closed-loop system is modeled as a linear system with decentralized dead-zone nonlinearity by incorporating the state estimation errors into the state equation as new state variables. The design results for both stability and stabilization of the closed-loop system are given in the form of matrix inequalities. However, the matrix inequality conditions to compute the design parameters are non-convex in general. Then by constructing an appropriate objective function in the form of Linear Matrix Inequality (LMI), the non-convex optimization problem is converted to unconstrained nonlinear optimization problem, which can be solved by any nonlinear programming algorithm. A numerical example illustrates the effectiveness of the proposed design technique.

Keywords: Fault-tolerant control, uncertain linear systems, actuator saturation, actuator faults, non-convex optimization, nonlinear programming.

I INTRODUCTION

In order to tolerate component malfunctions while maintaining desirable stability and performance properties, a significant amount of research on fault-tolerant control systems (FTCS) has been conducted in the past several decades [1]. However, many challenging issues still remain open for further research and development. One of the challenges is how to deal with actuator saturation in the presence of actuator faults. It is currently an active research topic in the literature (see, e.g., [2]).

All real-world applications of feedback control involve control actuators with amplitude and rate limitations. The control design techniques that ignore these actuator limits may degrade the closed-loop performance, and may even induce closed-loop instability. A great amount of attention has been focused on the stability and stabilization for systems with saturating actuators (see, e.g., [3]). The control problem of uncertain linear system with saturating actuators has also attracted considerable attention and various techniques have been developed (see, e.g., [4]).

However, to our knowledge, the fault-tolerant control problem considering simultaneously the uncertainties and actuator saturation has not been well addressed in the literature. This is the main motivation and topic to be studied in this paper. Besides, the bounded disturbances are also considered. Based on the classical Luenberger observer, the matrix inequality conditions are given to stabilize the overall system, and a nonlinear programming method is proposed to solve the non-convex optimization problem.

The paper is organized as follows. The problem to be treated is stated in Section II. Some preliminaries are given in Section III. Section IV presents a non-convex matrix inequality conditions to compute the observer gain and feedback control gain. The non-convex optimization problem is solved by nonlinear programming in Section V. The effectiveness of the proposed technique is illustrated by a numerical example in Section VI. Finally, a conclusion is drawn in Section VII.

Notations: \( X_i \) denotes the \( i \)-th row of a matrix \( X \), and \( X_{i,j} \) denotes the element located in its \( i \)-th row and \( j \)-th column. \( X^T \) denotes its transpose. Symmetric elements in the matrix are denoted by \( \ast \). \( \text{sym}(X) \) denotes \( X + X^T \). \( I_n \) denotes \( n \)-th order identity matrix. \( 0 \) denotes zero matrix with appropriate dimension.

II PROBLEM STATEMENT

Consider the following uncertain linear system subject to actuator saturation and faults described by

\[
\begin{align*}
\dot{x} &= (A_0 + \Delta A)x + BM\text{sat}(u) + E\omega \\
y &= (C_0 + \Delta C)x + D\omega
\end{align*}
\]

(1)

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control input vector, \( y \in \mathbb{R}^r \) is the output vector, \( \omega \in \mathbb{R} \) is a vector of
bounded disturbances belonging to the set \( \omega \in \{ \omega (t) \mid \omega (t) \leq 1, \forall t \geq 0 \} \). Matrices \( A_h, B, C_o, D, E \) are constant real matrices of appropriate dimensions. It is assumed that \( (A_h, B) \) is stabilizable and \( (A_h, C_o) \) is observable. Besides, the uncertainties \( \Delta A \) and \( \Delta C \) are defined as follows:

\[
\begin{align*}
\Delta A &= G_i F_i (t) E_i \\
\Delta C &= G_e F_e (t) E_c
\end{align*}
\]

(2)

where \( F_i (t) \), \( F_e (t) \) are matrices of appropriate dimensions. The saturating term in (1) can be defined as follows:

\[
\text{sat}(u) = \left[ \text{sat}(u_1), \text{sat}(u_2), \ldots, \text{sat}(u_m) \right]^T
\]

(3)

Matrix \( M \) represents the following actuator fault model:

\[
M = \text{diag} \{ m_1, m_2, \ldots, m_m \}
\]

(4)

Introduce the following matrices

\[
M_0 = \text{diag} \{ m_0_1, m_0_2, \ldots, m_0_m \}
\]

(5)

where \( m_0_i = \frac{1}{2} (\bar{m}_i + m_i) \), \( j_i = \frac{\bar{m}_i - m_i}{\bar{m}_i + m_i} \).

Then it can be obtained that

\[
M = M_0 (I + L), |L| \leq J \leq I
\]

(6)

Remark 1: \( m_0 = 0 \) means the total outage of the \( i \)-th actuator, while \( m_i = 1 \) denotes a healthy actuator. \( 0 \leq m_i \leq \bar{m}_i \) means partial loss of the \( i \)-th actuator.

Denote \( B_0 = BM_0, AB = BM, B = B_L \), then system (1) can be written as:

\[
\begin{align*}
\dot{x} &= (A_h + \Delta A) x + (B_0 + \Delta B) \text{sat}(u) + E \omega \\
y &= (C_o + \Delta C) x + D \omega
\end{align*}
\]

(7)

In this paper, a state-feedback control matrix \( K \) is designed to stabilize system (7), based on the Luenberger observer:

\[
\begin{align*}
\dot{x} &= A_h \dot{x} + B_0 u + L_0 (y - C_o \dot{x}) \\
u &= K \dot{x}
\end{align*}
\]

(8)

III PRELIMINARIES

The composite system can be written in the coordinates \( x \) and \( e = x - \hat{x} \) as the following equations:

\[
\begin{align*}
\dot{x} &= (A_h + \Delta A) x + (B_0 + \Delta B) \text{sat}(u) + E \omega \\
\dot{e} &= (A_o - L_0 C_o) e + (\Delta A - L_0 \Delta C) x \\
&+ B_0 \left[ \text{sat}(u) - u \right] + \Delta B \text{sat}(u) + (E - L_0 D) \omega
\end{align*}
\]

(9)

It follows that

\[
\dot{z} = A \hat{z} + B \text{sat} (u) + B_0 \left[ \text{sat} (u) - u \right] + E \omega
\]

(10)

where \( z = \begin{bmatrix} x \\ e \end{bmatrix} \), \( A = \begin{bmatrix} A_h & \Delta A \\ \Delta A & L_0 \Delta C \end{bmatrix}, B_1 = \begin{bmatrix} B_0 & \Delta B \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, E_o = \begin{bmatrix} E \\ E - L_0 D \end{bmatrix} \).

Define \( K_e = K \begin{bmatrix} I_{n_h} & -I_{n_h} \end{bmatrix} \), then the composite system can be written as:

\[
z = (A + B K_e) z - (B + B_1) \Psi(u) + E \omega
\]

(11)

where \( \Psi(u) = u - \text{sat}(u) \), \( u = K_e \hat{z} \).

Let \( Z_o \) be a set of admissible initial conditions, and the control objective is to find an observer-based state-feedback control law as shown in (8), such that all trajectories of system (11) starting from within \( Z_o \) will remain inside it, that is, \( Z_o \) is an invariant set for system (11). In this work, \( Z_o \) is considered as an ellipsoidal set defined as follows:

\[
\Omega(P) = \{ z \in \mathbb{R}^{2n} \mid z^T P z \leq 1, P > 0 \}
\]

(12)

To present the main results, the following lemmas are used.

Lemma 1 [5] \( \Psi^T (K_e z) T [\Psi^T (K_e z) - G z] \leq 0 \), for any diagonal and positive definite matrix \( T \), if \( z \in S(K_e - G, \epsilon) \), \( z = [z_1, \ldots, z_m] \), \( \epsilon = 1, 2, \ldots, m \).

Lemma 2 [6] Given matrices \( M, F, N \) with appropriate dimensions. If \( F^T F \leq I \), then \( \forall \epsilon > 0 \), the following inequality holds

\[
M N^T + N F^T M^T \leq \epsilon M M^T + \epsilon I N^T N
\]

(13)

Lemma 3 Consider a linear time-varying system

\[
z(t) = A(t) z(t)
\]

(14)

where \( A(t) \) is a piecewise continuous matrix-valued function.

If there exists a quadratic Lyapunov function

\[
V(z) = z^T(t) P z(t)
\]

(15)

such that \( \dot{V} < 0 \), then system (14) is uniformly asymptotically stable.

Proof: Since the right-hand side of (14) satisfies a global Lipschitz condition, then the solution of (14) is unique. Since

\[
\dot{V}(z) = z^T(t) [ \tilde{\Delta}^T(t) P + P A(t) z(t) ]
\]

(16)

then \( \dot{V}(z) < 0 \) is obtained by satisfying

\[
\tilde{\Delta}^T(t) P + P A(t) z(t) < 0
\]

(17)

which is sufficient to ensure the uniformly asymptotically stability of system (14) (for a proof see [7]), with the fact that the exponentially stability is equivalent to the uniformly asymptotically stability for system (14).

IV MAIN RESULTS

Theorem 1 If there exist matrices \( P \in \mathbb{R}^{2n \times 2n}, K_e \in \mathbb{R}^{2n \times 2n}, G \in \mathbb{R}^{n \times 2n}, \epsilon \in \mathbb{R}^{n \times 2n}, T \in \mathbb{R}^{n \times n} \), satisfying

\[
\begin{align*}
M N^T + N F^T M^T &\leq \epsilon M M^T + \epsilon I N^T N \\
\Psi^T (K_e z) T [\Psi^T (K_e z) - G z] &\leq 0
\end{align*}
\]

Then the composite system is uniformly asymptotically stable.
\[
P(A + B_t K_t) + (A + B_t K_t) P + PE_{\omega}^2 P + P^* - P (B_t + B_t) + G^T T - 2T < 0
\]

(18)

\[
\begin{bmatrix}
P & * \\
K_t - G_t & c_i^2
\end{bmatrix} \geq 0, i = 1, 2, \ldots, m
\]

(19)

then \( \Omega(P) \) is an invariant set for system (11).

**Proof:** Since \( z(t) \in S(K_t - G_t, \varepsilon) \) is equivalent to (19), then according to Lemma 1, \( \mathcal{V}(z) < 0 \) is equivalent to

\[
\mathcal{V}(z) = \dot{\mathcal{V}}(z) = -2 \Psi^T (K_t z) T \Psi(K_t z) + 2 \Psi^T (K_t z) T G z < 0
\]

(20)

For

\[
2z^T PE_{\omega} z \leq z^T PE_{\omega}^2 P z + \omega^T \omega \leq z^T PE_{\omega}^2 P z + 1
\]

(21)

then

\[
\begin{bmatrix}
\mathcal{V}(z) \\
\dot{\mathcal{V}}(z)
\end{bmatrix} \geq 0, i = 1, 2, \ldots, m
\]

(22)

where \( \Theta \) is the left-hand side of (18).

Observing that on the boundary of \( \Omega(P) \), \( z^T P z = 1 \). Hence, if (18) holds, then \( \mathcal{V}(z) < 0 \). Therefore, \( \Omega(P) \) is an invariant set.

To remove the uncertainties in matrix \( A \) and \( B_t \), the following theorem can be used.

**Theorem 2** If there exist matrices \( X, Y \in \mathbb{R}^{m \times n} \), \( L_0 \in \mathbb{R}^{m \times p} \), a symmetric positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), diagonal positive definite matrices \( H \in \mathbb{R}^{m \times m} \), positive scalars \( \varepsilon_i > 0, i = 1, 2, 3 \) satisfying

\[
\begin{bmatrix}
\Phi & * \\
* & -\varepsilon_i I_n
\end{bmatrix} \geq 0, i = 1, 2, \ldots, m
\]

(23)

\[
[Q, Y - X, c_i^2] \geq 0, i = 1, 2, \ldots, m
\]

(24)

where

\[
\Phi = \begin{bmatrix} A_0 & 0 \\ 0 & -2H \end{bmatrix}
\]

(25)

\[
\Phi = \begin{bmatrix} Q + QA^T + BY^T + YB_i^T + E_i E_i^T + Q & -(B_t + B_t) H + X^T \\ * & * \end{bmatrix} < 0
\]

(26)

By Lemma 2, \( \forall \varepsilon_i > 0, i = 1, 2, 3 \), it follows that

\[
\Phi = \begin{bmatrix} Q + QA^T + BY^T + YB_i^T + E_i E_i^T + Q & -(B_t + B_t) H + X^T \\ * & * \end{bmatrix} < 0
\]

(27)

V NONLINEAR PROGRAMMING

Due to the multiplication items about the decision variables, Eq. (23) is not a LMI in general. However, for some given \( L_0 \), (27) is a MAXDET (Determinant Maximization) problem with LMI constraints, which is not difficult to solve.

In this paper, the nonlinear programming is used to search the \( L_0 \) space. The optimization problem (27) can be converted into the unconstrained nonlinear optimization problem by defining a nonlinear scalar function as follows:

\[
\min_{s.t. (23)} \text{log det } Q
\]

(28)

\[
f(L_0) = \text{log det } Q
\]

For computation, a popular direct search method called Nelder-Mead simplex algorithm will be used, which has been realized in a MATLAB function ‘fminsearch’.

VI ILLUSTRATIVE EXAMPLE

The presented design methods will be demonstrated in the following example. The uncertain plant is defined by

\[
A_0 = \begin{bmatrix} 0 & 1 \\ -10 & -3 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_0 = \begin{bmatrix} 20 & 0 \end{bmatrix}, D = 1,
\]

\[
E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}, G_c = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, E_A = E_c = 1.
\]
Suppose the actuator faults to be tolerated are such that \( \bar{n} = 1, \bar{m} = 0.8 \), and the control constraint is given by \( \bar{u} = 1 \).

Take \( L_0 = [1 \ 1]^T \) as the initial value, and solve the optimization problem (27) by unconstrained nonlinear programming, it is obtained that \( L_0^* = [0.5027 \ 1.1068] \), \( \gamma^* = -99.8063 \), and \( K^* = [-0.0536 \ -0.0240 \ 0.3082 \ 0.0795] \). The searching process of optimal \( L_0 \) is shown in Fig. 1.

To see the responses of the closed-loop system, it is assumed

\[
F_A = \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(2t) \end{bmatrix}, \quad F_c = \begin{bmatrix} \sin(2t) & 0 \\ 0 & \sin(3t) \end{bmatrix}, \quad \omega(t) = \sin(t),
\]

and the actuator was damaged at \( t_f = 0.2s \) with remaining effectiveness of 50%.

Given the initial state \( z_0 = [100 \ 100 \ 50 \ -50] \) in the invariant ellipsoid \( \Omega(P) \) with non-zero observation errors, by running the composite system (11) with the optimal \( L_0^* \) and gain \( K^* \), the responses of the states and control input can be obtained as shown in Fig. 2. It is obvious that the system states including the observation errors can converge to zero, although the actuator is saturated in the initial phase.

![Fig. 1 Variation of \( \gamma \)](image1)

![Fig. 2 Response of states and control with non-zero initial observation errors](image2)

## VII CONCLUSION

Based on the classical Luenberger observer to estimate the state variables and modeling the actuator fault as an uncertainty in the input matrix, a closed-loop system with decentralized dead-zone nonlinearity is obtained by incorporating the state estimation errors into the state equation as new state variables. Then, the matrix inequality conditions are obtained for stabilization of the uncertain linear systems subject to actuator faults, saturation and disturbances. An effective nonlinear programming method is proposed to solve the optimization problem by constructing an appropriate nonlinear function. An example is presented for illustration. For given continuous actuator fault, the obtained optimal observer gain and controller gain guarantee good performances of the closed-loop system.

## ACKNOWLEDGMENTS

The work reported in this paper is partially sponsored by China Scholarship Council (CSC) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

## REFERENCES


