

SOLUTION GUIDELINES TO FINAL EXAM

Problem 1

- (a)
- (i) Speed of response: Rise time; Peak time; Natural frequency
 - (ii) Relative stability: Percentage overshoot; 2% settling time; Dominant pole location

Rise time: The time for a step response to pass through desired value (or, 90% of the desired value) for the first time

Peak time: The time for a step response to reach the first peak value.

Natural frequency: the frequency at which the system oscillates when excited by an initial-condition change.

Percentage overshoot = $100[\text{First peak value} - \text{Desired value}]/[\text{Desired value}]$

2% settling time: The time for a step response to settle within $\pm 2\%$ of the desired value.

Dominant pole location: The distance of the pole that is closest to the origin on the s -plane.

- (b)
- (i) The feedback transfer function may represent a position plus velocity (or, tacho feedback) controller or a proportional plus derivative (PD) controller.

(ii)
Since the system is 3rd order and Type 2, the forward transfer function may be expressed as

$$G(s) = \frac{K}{s^2 (s + a)}$$

Then with the given feedback transfer function $H(s)$, the closed-loop transfer function is

$$\tilde{G}(s) = \frac{\frac{K}{s^2(s+a)}}{\left[1 + \frac{K(1+bs)}{s^2(s+a)}\right]} = \frac{K}{[s^2(s+a) + K(1+bs)]} = \frac{K}{[s^3 + as^2 + Kbs + K]} \quad (i)$$

This may be expressed in the form

$$\tilde{G} = \frac{K}{(s+p)(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad (ii)$$

with

$$p = 10\zeta\omega_n \quad (iii)$$

Now,

$$T_s = \frac{4}{3} \text{ seconds} = \frac{4}{\zeta\omega_n} \quad \text{for low damping}$$

$$T_p = \frac{\pi}{4} \text{ seconds} = \frac{\pi}{\omega_d}$$

Hence,

$$\zeta\omega_n = 3$$

$$\omega_d = 4 = \sqrt{1 - \zeta^2} \omega_n^2 = \sqrt{\omega_n^2 - \zeta^2 \omega_n^2} = \sqrt{\omega_n^2 - 3^2}$$

We have,

$$\omega_n = 5, \quad \omega_d = 4, \quad \zeta = 0.6$$

From (iii): $p = 10\zeta\omega_n = 10 \times 3 = 30$

Then, (ii) becomes

$$\tilde{G} = \frac{K}{(s+30)(s^2 + 6s + 25)} = \frac{K}{(s^3 + 36s^2 + 205s + 750)}$$

Compare this denominator with that of (i):

We have:

$$a = 36$$

$$K = 750$$

$$Kb = 205$$

Hence,

$$b = 205/750 = 41/150$$

Then, the forward transfer function is completely determined:

$$G(s) = \frac{750}{s^2(s+36)}$$

(iii) As obtained before,

$$\zeta = 0.6$$

The % overshoot is given by

$$\begin{aligned} PO &= 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \\ &= 100 \exp\left(-\frac{\pi \times 0.6}{\sqrt{1-0.6^2}}\right) \\ &= 100 \exp(-3\pi/4) \end{aligned}$$

(iv) Since the system is Type 2, the steady state error to a ramp input will be zero.

Problem 2

(a)

- (i) Stable, because all four poles are on the LHP.
- (ii) Unstable, because there is one pole on the RHP.
- (iii) Marginally stable, because there are two imaginary poles (on the imaginary axis), which will generate constant-amplitude oscillations when given an initial excitation. The remaining three poles are on the LHP, which will generate decaying responses.
- (iv) Unstable, because there is a pair of complex conjugate poles on the RHP.
- (v) Unstable, because the two poles at the origin will generate a linearly growing response when excited by an initial condition. The remaining three poles are on the LHP, which will produce decaying responses.

(b) The system characteristic equation is

$$s^3 + 12s^2 + 61s + 150 = 0$$

(i)

Routh Array:

s^3	1	61
s^2	12	150
s^1	$\frac{12 \times 61 - 1 \times 150}{12} = 48.5$	0
s^0	150	

Since the entries of the first column are all positive, there are no unstable poles. The system is stable.

(ii)

Denote the shifted poles by \tilde{s}

We have $\tilde{s} = s + 3$ or $s = \tilde{s} - 3$

Substitute in the original characteristic equation. The characteristic equation of the modified system (with shifted poles) is:

$$(\tilde{s} - 3)^3 + 12(\tilde{s} - 3)^2 + 61(\tilde{s} - 3) + 150 = 0$$

or, $\tilde{s}^3 - 9\tilde{s}^2 + 27\tilde{s} - 27 + 12\tilde{s}^2 - 72\tilde{s} + 108 + 61\tilde{s} - 183 + 150 = 0$

or, $\tilde{s}^3 + 3\tilde{s}^2 + 16\tilde{s} + 48 = 0$

Routh array for the modified system:

\tilde{s}^3	1	16
\tilde{s}^2	3	48
\tilde{s}^1	$\frac{3 \times 16 - 1 \times 40}{3} = 0$	0
\tilde{s}^0	?	

Notice the row with all zero entries. This indicates a marginally stable system, and gives the auxiliary equation (from the row right above the zero row):

$$3\tilde{s}^2 + 48 = 0$$

The corresponding pair of imaginary poles is $\pm j4$

Let the remaining pole (of the modified system) be at $-p$. Then, the characteristic polynomial of the modified system is

$$(\tilde{s}^2 + 16)(s + p) = \tilde{s}^3 + 3\tilde{s}^2 + 16\tilde{s} + 48$$

By comparing the coefficients on the two sides we have

$$16p = 48$$

Or,

$$p = 3$$

This pole is on the LHP (i.e., stable). Hence the overall system is marginally stable (not unstable), in view of the purely imaginary pair of poles

(iii)

As found in Part (ii), the poles of the modified (shifted) system are:

$$\pm j4 \text{ and } -3$$

The poles of the original system are obtained by shifting these poles back to the left by 3. The resulting poles are:

$$-3 \pm j4 \text{ and } -6$$

This further confirms that the original system is stable.

Problem 3

The system transfer function $G(s)$ may be expressed as

$$G(s) = \frac{K(s^2 + b)}{(s^3 + a_2s^2 + a_1s + a_0)}$$

Note 1: Even though, generally, we would need six parameters, for this problem we need only five parameters, in view of the information given under Item 3 of the problem, as will be clear.

Note 2: If a pole or a zero is complex, it has to occur in complex conjugates.

Since the product of the three poles is -4, this must be equal to the negative of the constant term in the characteristic polynomial. We have,

$$a_0 = 4$$

The frequency transfer function of the system is

$$G(j\omega) = \frac{K(b - \omega^2)}{(a_0 - a_2\omega^2) + j\omega(a_1 - \omega^2)} \quad (i)$$

When the system is excited at frequency $\omega = 4$, the output at steady state = 0.

Hence, the numerator of (i) must vanish at this frequency:

$$b - \omega^2 = 0 = b - 4^2$$

We have

$$b = 16$$

When the system is excited at frequency $\omega = 2$, since the output at steady state has a phase lag of 180° , the imaginary term in the denominator of (i) must vanish. Hence,

$$a_1 - \omega^2 = 0 = a_1 - 2^2$$

We have

$$a_1 = 4$$

When the system is excited at frequency $\omega = \sqrt{2}$, the output at steady state has a phase lag of 90° , the real term in the denominator of (i) must vanish. Hence,

$$a_0 - a_2\omega^2 = 0 = a_0 - 2a_2$$

We have

$$a_0 = 2a_2$$

Or,

$$a_2 = a_0 / 2 = 4 / 2 = 2$$

From (i), the DC gain of the system is

$$G(0) = \frac{Kb}{a_0} = 8$$

Hence,

$$K = \frac{8a_0}{b} = \frac{8 \times 4}{16} = 2$$

Then, the transfer function is

$$G(s) = \frac{2(s^2 + 16)}{(s^3 + 2s^2 + 4s + 4)} \quad \text{or} \quad G(j\omega) = \frac{2(16 - \omega^2)}{(4 - 2\omega^2) + j\omega(4 - \omega^2)}$$

Note 1: From (i) it can be confirmed now that at frequency $\omega = 2$, the denominator is real and negative and the numerator is real and positive, giving a real and negative frequency response function $G(j\omega)$, which corresponds to a phase lag of 180° .

Note 2: Using Routh-Hurwitz it can be shown that the system is stable.