Appendix A: Transform Techniques

Many people use "transforms" without even knowing it. A "transform" is simply a number, variable, or function in a different form. For example, since $10^2 = 100$, you can use the exponent (2) to represent the number 100. Doing this for all numbers (i.e., using their exponent to the base 10), results in a "table of logarithms." One can perform mathematical computations using only "logarithms." The logarithm transforms all numbers into their exponential equivalents; a table of such transforms (i.e., a log table) enables a user to quickly transform any number into its exponent, do the computations using exponents (where, a product becomes an addition and a division becomes a subtraction), and transform the result back (i.e., inverse logarithm) into the original form. It is seen that the computations have become simpler by using logarithms, but at the cost of the time and effort for transformation and inverse transformation.

Other common transforms include the Laplace Transform, Fourier Transform, and Z-transform. In particular, the Laplace Transform provides a simple, algebraic way to solve (i.e., integrate) a linear differential equation. Most functions that we use are of the form t^n, $\sin \omega t$, or e^t, or some combination of them. Thus, in the expression

$$y = f(t)$$

the function y is quite likely a power, a sine, or an exponential function. Also, often, we have to work with derivatives and integrals of these functions, and differential equations containing these functions. These tasks can be greatly simplified by the use of the Laplace transform.

Concepts of frequency-response analysis originate from the nature of the response of a dynamic system to a sinusoidal (i.e., harmonic) excitation. These concepts can be generalized because the time-domain analysis, where the independent variable is time (t) and the frequency-domain analysis, where the independent variable is frequency (ω) are linked through the Fourier transformation. Analytically, it is more general and versatile to use the Laplace transformation, where the independent variable is the Laplace variable (s) which is complex (nonreal). This is true because analytical Laplace transforms may exist even for time functions that do not have "analytical" Fourier transforms. But with compatible definitions, the Fourier transform results can be obtained form the Laplace transform results simply by setting $s = j\omega$. In the present appendix we will formally introduce the Laplace transformation and the Fourier transformation, and will illustrate how these techniques are useful in the analysis of dynamic systems. The preference of one domain over another will depend on such factors as the nature of the excitation input, the type of the analytical model available, the time duration of interest, and the quantities that need to be determined.

A.1 Laplace Transform

The Laplace transformation relates the time domain to the *Laplace domain* (also called *s-domain* or complex frequency domain). The Laplace transform $Y(s)$ of a piecewise-continuous function or signal $y(t)$ is given, by definition, as

$$Y(s) = \int_0^{\infty} y(t)\exp(-st)\,dt \tag{A.1}$$

and is denoted using the Laplace operator $\mathcal{L}$, as

$$Y(s) = \mathcal{L}y(t) \tag*{(A.1)*}$$

Here, s is a complex independent variable known as the Laplace variable, defined by

$$s = \sigma + j\omega \tag{A.2}$$

where, σ is a real-valued constant that will make the transform (Equation A.1) finite, ω is simply frequency, and $j = \sqrt{-1}$. The real value (a) can be chosen sufficiently large so that the integral in Equation A.1 is finite even when the integral of the signal itself (i.e., $\int y(t)dt$) is not finite. This is the reason why, for example, Laplace transform is better behaved than Fourier transform, which will be defined later, from the analytical point of view. The symbol s can be considered to be a constant, when integrating with respect to t, in Equation A.1.

The inverse relation (i.e., obtaining y from its Laplace transform) is

$$y(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} Y(s)\exp(st)\,ds \tag{A.3}$$

and is denoted using the inverse Laplace operator $\mathcal{L}^{-1}$, as

$$y(t) = \mathcal{L}^{-1}Y(s) \tag*{(A.3)*}$$

The integration in Equation A.3 is performed along a vertical line parallel to the imaginary (vertical) axis, located at σ from the origin in the complex Laplace plane (s-plane). For a given piecewise-continuous function $y(t)$, the Laplace transform exists if the integral in Equation A.1 converges. A sufficient condition for this is

$$\int_0^{\infty} |y(t)|\exp(-\sigma t)dt < \infty \tag{A.4}$$

Convergence is guaranteed by choosing a sufficiently large and positive σ. This property is an advantage of the Laplace transformation over the Fourier transformation.

A.1.1 Laplace Transforms of Some Common Functions

Now we determine the Laplace transform of some useful functions using Equation A.1. Usually, however, we use Laplace transform tables to obtain these results.

A.1.1.1 Laplace Transform of a Constant

Suppose our function $y(t)$ is a constant, B. Then the Laplace transform is:

$$\mathcal{L}(B) = Y(s) = \int_0^\infty Be^{-st}dt$$

$$= B\frac{e^{-st}}{-s}\bigg|_0^\infty = \frac{B}{s}$$

A.1.1.2 Laplace Transform of the Exponential

If $y(t)$ is e^{at}, its Laplace transform is

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st}e^{at}dt$$

$$= \int_0^\infty e^{(a-s)t}dt$$

$$= \frac{1}{(a-s)}e^{(a-s)t}\bigg|_0^\infty = \frac{1}{s-a}$$

Note: If $y(t)$ is e^{-at}, it is obvious that the Laplace transform is

$$\mathcal{L}(e^{-at}) = \int_0^\infty e^{-st}e^{-at}dt$$

$$= \int_0^\infty e^{-(a+s)t}dt$$

$$= \frac{-1}{(a-s)}e^{-(a+s)t}\bigg|_0^\infty = \frac{1}{s+a}$$

This result can be obtained from the previous result simply by replacing a with $-a$.

A.1.1.3 Laplace Transform of Sine and Cosine

In the following, the letter $j = \sqrt{-1}$. If $y(t)$ is $\sin \omega t$, the Laplace transform is

$$\mathcal{L}(\sin\omega t) = \int_0^\infty e^{-st}(\sin\omega t)dt$$

Consider the identities:

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

If we add and subtract these two equations, respectively, we obtain the expressions for the sine and the cosine in terms of $e^{j\omega t}$ and $e^{-j\omega t}$:

$$\cos \omega t = \frac{1}{2}\left(e^{j\omega t} + e^{-jwt}\right)$$

$$\sin \omega t = \frac{1}{2i}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$\mathcal{L}(\cos \omega t) = \frac{1}{2} L\left(e^{j\omega t}\right) + \frac{1}{2} L\left(e^{-j\omega t}\right)$$

$$\mathcal{L}(\sin \omega t) = \frac{1}{2} L\left(e^{j\omega t}\right) - \frac{1}{2} L\left(e^{-j\omega t}\right)$$

We have just seen that

$$\mathcal{L}\left(e^{at}\right) = \frac{1}{s-a}; \; \mathcal{L}\left(e^{-at}\right) = \frac{1}{s+a}$$

Hence,

$$\mathcal{L}\left(e^{j\omega t}\right) = \frac{1}{s - j\omega t}; \; \mathcal{L}\left(e^{-j\omega t}\right) = \frac{1}{s + j\omega t}$$

Substituting these expressions, we get

$$\begin{aligned} \mathcal{L}(\cos \omega t) &= \frac{1}{2}\left[\frac{1}{s - j\omega}\right] + \frac{1}{2}\left[\frac{1}{s + j\omega}\right] \\ &= \frac{1}{2}\left[\frac{s + j\omega}{s^2 - (j\omega)^2} + \frac{s - j\omega}{s^2 - (j\omega)^2}\right] \\ &= \frac{s}{s^2 + \omega^2} \end{aligned}$$

$$\mathcal{L}(\sin\omega t) = \frac{1}{2j}L\left(e^{j\omega t} - e^{-j\omega t}\right)$$
$$= \frac{1}{2j}\left[\frac{1}{s - j\omega}\right] - \frac{1}{2j}\left[\frac{1}{s + j\omega}\right]$$
$$= \frac{1}{2j}\left[\frac{s + j\omega}{s^2 - (j\omega)^2} + \frac{s - j\omega}{s^2 - (j\omega)^2}\right]$$
$$= \frac{1}{2j}\left[\frac{2j\omega}{s^2 + \omega^2}\right]$$
$$= \frac{\omega}{s^2 + \omega^2}$$

A.1.1.4 Laplace Transform of a Derivative

Let us transform a derivative of a function. Specifically, the derivative of a function y of t is denoted by $\dot{y} = (dy/dt)$. Its Laplace transform is given by

$$\mathcal{L}(\dot{y}) = \int_0^\infty e^{-st}\dot{y}dt = \int_0^\infty e^{-st}\frac{dy}{dt}dt \tag{A.5}$$

Now we integrate by parts, to eliminate the derivative within the integrand.
Integration by Parts: From calculus we know that $d(uv) = udv + vdu$

By integrating we get $uv = \int udv + \int vdu$

Hence,

$$\int udv = uv - vdu \tag{A.6}$$

This is known as integration by parts.

In Equation A.5, let

$$u = e^{-st} \text{ and } v = y$$

Then, $dv = dy = \frac{dy}{dt}dt = \dot{y}dt$

$$du = \frac{du}{dt}dt = -se^{-st}dt.$$

Substitute in Equation A.5 to integrate by parts:

$$\mathcal{L}(\dot{y}) = \int_0^\infty e^{-st} dy$$

$$= \int u dv = uv - \int v du$$

$$= e^{-st} y(t)\Big|_0^\infty - \int_0^\infty -se^{-st} y(t) dt$$

$$= -y(0) + s\mathcal{L}[y(t)]$$

$$= s\mathcal{L}(y) - y(0)$$

where $y(0)$ = initial value of y. This says that the Laplace transform of a first derivative $\dot{y}$, equals s times the Laplace transform of the function y minus the initial value of the function (the initial condition).

Note: We can determine the Laplace transforms of the second and higher derivatives by repeated application this result, for the first derivative. For example, the transform of the second derivative is given by

$$\mathcal{L}[\ddot{y}(t)] = \mathcal{L}\left[\frac{d\dot{y}(t)}{dt}\right] = s\mathcal{L}[\dot{y}(t)] - \dot{y}(0) = s\{s\mathcal{L}[y(t)] - y(0)\} - \dot{y}(0)$$

$$\text{or, } \mathcal{L}[\ddot{y}(t)] = s^2\mathcal{L}[y(t)] - sy(0) - \dot{y}(0)$$

A.1.2 Table of Laplace Transforms

Table A.1 shows the Laplace Transforms of some common functions. Specifically, the table lists functions as $y(t)$, and their Laplace transforms (on the right) as $Y(s)$ or $\mathcal{L}y(t)$. If one is given a function, one can get its Laplace transform from the table. Conversely, if one is given the transform, one can get the function from the table.

Some general properties and results of the Laplace transform are given in Table A.2.

In particular, note that, with zero initial conditions, differentiation can be interpreted as multiplication by s. Also, integration can be interpreted as division by s.

A.2 Response Analysis

The Laplace transform method can be used in the response analysis of dynamic systems, mechatronic and control systems in particular. We will give examples for the approach.

TABLE A.1

Laplace Transform Pairs

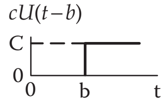
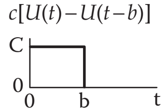
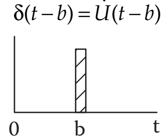
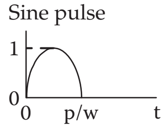
$y(t) = \mathcal{L}^{-1}[Y(s)]$	$\mathcal{L}[y(t)] = Y(s)$
B	B/s
e^{-at}	$1/(s+a)$
e^{at}	$1/(s-a)$
Sinh at	$a/(s^2-a^2)$
cosh at	$s/(s^2-a^2)$
sin ωt	$\omega/(s^2+\omega^2)$
cos ωt	$s/(s^2+\omega^2)$
e^{-at} sin ωt	$\omega/((s+a)^2+\omega^2)$
e^{-at} cos ωt	$s+a/((s+a)^2+\omega^2)$
Ramp t	$1/s^2$
$e^{-at}(1-at)$	$s/(s+a)^2$
$y(t)$	$Y(s)$
$(dy/dt) = \dot{y}$	$sY(s) - y(0)$
$(d^2y/dt^2) = \ddot{y}$	$s^2Y(s) - sy(0) - \dot{y}(0)$
$(d^3y/dt^3) = \dddot{y}$	$s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0)$
$\int_a^t y(t)dt$	$\frac{1}{s}Y(s) - \frac{1}{s}\int_0^a y(t)dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
Unit step $U(t) = 1$ for $t \geq 0$ $= 0$ otherwise	$1/s$
Delayed step $cU(t-b)$ 	$\frac{c}{s}e^{-bs}$
Pulse $c[U(t)-U(t-b)]$ 	$c\left(\frac{1-e^{-bs}}{s}\right)$
Impulse function $\delta(t)$	1
Delayed impulse $\delta(t-b) = \dot{U}(t-b)$ 	e^{-bs}
Sine pulse 	$\left(\frac{\omega}{s^2+\omega^2}\right)\left(1+e^{-(\pi s/\omega)}\right)$

TABLE A.2
Important Laplace Transform Relations

$\mathcal{L}^{-1}F(s) = f(t)$	$\mathcal{L}f(t) = F(s)$
$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\exp(st)ds$	$\int_0^{\infty} f(t)\exp(-st)dt$
$k_1f_1(t) + k_2f_2(t)$	$k_1F_1(s) + k_2F_2(s)$
$\exp(-at)\,f(t)$	$F(s + a)$
$f(t-\tau)$	$\exp(-\tau s)F(s)$
$f^{(n)}(t) = \frac{d^n f(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f^1(0^+) - \cdots - f^{n-1}(0^+)$
$\int_{-\infty}^{t} f(t)dt$	$\frac{F(s)}{s} + \frac{\int_{-\infty}^{0} f(t)dt}{s}$
t^n	$\frac{n!}{s^{n+1}}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$

Example A.1

The capacitor-charge equation of the *RC* circuit shown in Figure A.1 is

$$e = iR + v \qquad \text{(i)}$$

$$\text{For the capacitor, } i = C\frac{dv}{dt} \qquad \text{(ii)}$$

Substitute Equation (ii) in Equation (i) to get the circuit equation:

$$e = RC\frac{dv}{dt} + v \qquad \text{(iii)}$$

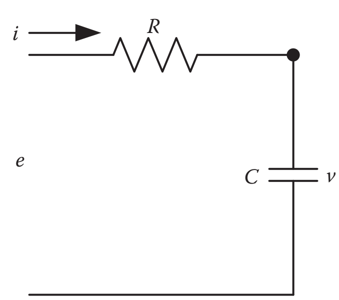


FIGURE A.1
An *RC* circuit with applied voltage *e* and voltage *v* across capacitor.

Take the Laplace transform of each term in Equation (iii), with all initial conditions = 0:

$$E(s) = RCsV(s) + V(s)$$

The transfer function expressed as the output–input ratio (in the transform form) is:

$$\frac{V(s)}{E(s)} = \frac{V(s)}{sRCV(s) + V(s)} = \frac{1}{sRC + 1} = \frac{1}{\tau s + 1} \quad \text{(iv)}$$

where $\tau = RC$.

The actual response can now be found from Table A.1 for a given input E. The first step is to get the transform into proper form (like Line 2):

$$\frac{1}{\tau s + 1} = \frac{1/\tau}{s + (1/\tau)} = \frac{a}{s + a} = a\left(\frac{1}{s + a}\right)$$

where $a = 1/\tau$. Suppose that input (excitation) e is a unit impulse. Its Laplace transform (see Table A.1) is $E = 1$. Then from Equation (iv),

$$V(s) = \frac{1}{\tau s + 1}$$

From Line 2 of Table A.1, the response is

$$v = ae^{-at} = \frac{1}{\tau}e^{-t/\tau} = \frac{1}{RC}e^{-t/RC}$$

A common transfer function for an overdamped second-order system (e.g., one with two RC circuit components of Figure A.1) would be

$$\frac{V(s)}{E(s)} = \frac{1}{(1 + \tau_1 s)(1 + \tau_2 s)}$$

This can be expressed as "partial fractions" in the from

$$\frac{A}{1 + \tau_1 s} + \frac{B}{1 + \tau_2 s}$$

and solved in the usual manner.

Example A.2

The transfer function of a thermal system is given by

$$G(s) = \frac{2}{(s + 1)(s + 3)}$$

If a unit step input is applied to the system, with zero initial conditions, what is the resulting response?

Solution

$$\text{Input } U(s)=\frac{1}{s} \text{ (for a unit step)}$$

$$\text{Since } \frac{Y(s)}{U(s)}=\frac{2}{(s+1)(s+3)}$$

the output (response)

$$Y(s)=\frac{2}{s(s+1)(s+3)}$$

Its inverse Laplace transform gives the time response. For this, first convert the expression into partial fractions as

$$\frac{2}{s(s+1)(s+3)}=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s+3)} \tag{i}$$

The unknown A is determined by multiplying Equation (i) throughout by s and then setting $s=0$. We get

$$A=\frac{2}{(0+1)(0+3)}=\frac{2}{3}$$

Similarly, B is obtained by multiplying Equation (i) throughout by $(s+1)$ and then setting $s=-1$. We get

$$B=\frac{2}{(-1)(-1+3)}=-1$$

Next, C is obtained by multiplying Equation (i) throughout by $(s+3)$ and then setting $s=-3$. We get

$$C=\frac{2}{(-3)(-3+1)}=\frac{1}{3}$$

Hence,

$$Y(s)=\frac{2}{3s}-\frac{1}{(s+1)}+\frac{1}{3(s+3)}$$

Take the inverse transform using Line 2 of Table A.1.

$$y(t) = \frac{2}{3} - e^{-t} + \frac{1}{3}e^{-3t}$$

Example A.3

The transfer function of a damped simple oscillator is known to be of the form

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

where ω_n = undamped natural frequency; ζ = damping ratio.

Suppose that a unit step input (i.e., $U(s) = (1/s)$) is applied to the system. Using Laplace transform tables determine the resulting response, with zero initial conditions.

Solution

$$Y(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

The corresponding partial fractions are of the form

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \quad \text{(i)}$$

We need to determine A, B, and C.

Multiply Equation (i) throughout by s and set $s = 0$. We get

$$A = 1$$

Next note that the roots of the characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

are

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n = -\zeta\omega_n \pm j\omega_d$$

These are the poles of the system and are complex conjugates. Two equations for B and C are obtained by multiplying Equation (i) by $s + \zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n$ and setting $s = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n$ and by multiplying Equation (i) by $s + \zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n$ and setting $s = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n$. We obtain $B = -1$ and $C = -2\zeta\omega_n$. Consequently,

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{\left[(s + \zeta\omega_n)^2 + \omega_d^2\right]} - \frac{\zeta}{\sqrt{1 - \zeta^2}} \cdot \frac{\omega_d}{\left[(s + \zeta\omega_n)^2 + \omega_d^2\right]} \end{aligned}$$

where, $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ = damped natural frequency.
Now use Table A.1 to obtain the inverse Laplace transform:

$$y_{step}(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)$$

where, $\cos\phi = \zeta$ = damping ratio; $\sin\phi = \sqrt{1-\zeta^2}$.

Example A.4

The open-loop response of a plant to a unit impulse input, with zero initial conditions, was found to be $2e^{-t}\sin t$. What is the transfer function of the plant?

Solution

By linearity, since a unit impulse is the derivative of a unit step, the response to a unit impulse is given by the derivative of the result given in the previous example; thus

$$y_{impulse}(t) = \frac{\zeta\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) - \frac{\omega_d}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_d t + \phi)$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\cos\phi\sin(\omega_d t + \phi) - \sin\phi\cos(\omega_d t + \phi)\right]$$

or

$$y_{impulse}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

Compare this with the given expression. We have

$$\frac{\omega_n}{\sqrt{1-\zeta^2}} = 2;\ \zeta\omega_n = 1;\ \ \omega_d = 1$$

But,

$$\omega_n^2 = (\zeta\omega_n)^2 + \omega_d^2 = 1 + 1 = 2$$

Hence

$$\omega_n = \sqrt{2}$$

Hence

$$\zeta = \frac{1}{\sqrt{2}}$$

The system transfer function is:

$$\frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{2}{s^2 + 2s + 2}$$

Example A.5

Express the Laplace transformed expression

$$X(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

as partial fractions. From the result, determine the inverse Laplace function $x(t)$.

Solution

$$X(s) = s + 2 + \frac{2}{s+1} - \frac{1}{s+2}$$

From Table A.1, we get the inverse Laplace transform

$$x(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t}$$

where $\delta(t)$ = unit impulse function.

A.3 Transfer Function

By the use of Laplace transformation, a *convolution integral* equation can be converted into an algebraic relationship. To illustrate this, consider the convolution integral which gives the response $y(t)$ of a dynamic system to an excitation input $u(t)$, with zero initial conditions. By definition Equation A.1, its Laplace transform, is written as

$$Y(s) = \int_0^\infty \int_0^\infty h(\tau)u(t-\tau)d\tau \exp(-st)dt \tag{A.7}$$

Note that $h(t)$ is the *impulse-response function* of the system. Since the integration with respect to t is performed while keeping τ constant, we have $dt = d(t - \tau)$. Consequently,

$$Y(s) = \int_{-\tau}^{\infty} u(t-\tau)\exp[-s(t-\tau)]\, d(t-\tau) \int_{0}^{\infty} h(\tau)\exp(-s\tau)d\tau$$

The lower limit of the first integration can be made equal to zero, in view of the fact that $u(t) = 0$ for $t < 0$. Again, by using the definition of Laplace transformation, the foregoing relation can be expressed as

$$Y(s) = H(s)U(s) \tag{A.8}$$

in which

$$H(s) = \mathcal{L}h(t) = \int_{0}^{\infty} h(t)\exp(-st)dt \tag{A.9}$$

Note that, by definition, the transfer function of a system, denoted by $H(s)$, is given by Equation A.8. More specifically, system transfer function is given by the ratio of the Laplace-transformed output and the Laplace-transformed input, with zero initial conditions. In view of Equation A.9, it is clear that the system transfer function can be expressed as the Laplace transform of the impulse-response function of the system. Transfer function of a linear and constant-parameter system is a unique function that completely represents the system. A physically realizable, linear, constant-parameter system possesses a unique transfer function, even if the Laplace transforms of a particular input and the corresponding output do not exist. This is clear from the fact that the transfer function is a system model and does not depend on the system input itself.

Note: The transfer function is also commonly denoted by $G(s)$. But in the present context we use $H(s)$ in view of its relation to $h(t)$.

Consider the nth-order linear, constant-parameter dynamic system given by

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_0 u + b_1 \frac{du(t)}{dt} + \cdots + b_m \frac{d^m u(t)}{dt^m} \tag{A.10}$$

For a physically realizable system, $m \le n$. By applying Laplace transformation and then integrating by parts, it may be verified that

$$\mathrm{L}\frac{d^k f(t)}{dt^k} = s^k \hat{F}(s) - s^{k-1}f(0) - s^{k-2}\frac{df(0)}{dt} - \cdots + \frac{d^{k-1}f(0)}{dt^{k-1}} \tag{A.11}$$

By definition, the initial conditions are set to zero in obtaining the transfer function. This results in

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots a_n s^n} \tag{A.12}$$

for $m \le n$. Note that Equation A.12 contains all the information that is contained in Equation A.10. Consequently, transfer function is an analytical model of a system. The transfer function may be employed to determine the total response of a system for a given input, even though it is defined in terms of the response under zero initial conditions. This is quite logical because the analytical model of a system is independent of the initial conditions of the system.

A.4 Fourier Transform

The Fourier transform $Y(f)$ of a signal $y(t)$ relates the time domain to the frequency domain. Specifically,

$$\begin{aligned} Y(f) &= \int_{-\infty}^{+\infty} y(t)\exp(-j2\pi ft)dt \\ &= \int_{-\infty}^{+\infty} y(t)e^{-\omega t}dt \end{aligned} \tag{A.13}$$

Using the Fourier operator "$\mathcal{F}$" terminology:

$$Y(f) = \mathcal{F}\ y(t) \tag{A.14}$$

Note that if $y(t)=0$ for $t < 0$, as in the conventional definition of system excitations and responses, the Fourier transform is obtained from the Laplace transform by simply changing the variable according to $s=j2\pi f$ or $s=j\omega$. The Fourier is a special case of the Laplace, where, in Equation A.2, $\sigma=0$:

$$Y(f) = Y(s)\big|_{s=j2\pi f} \tag{A.15}$$

or

$$Y(\omega) = Y(s)\big|_{s=j\omega} \tag{A.16}$$

The (complex) function $Y(f)$ is also termed the (continuous) *Fourier spectrum* of the (real) signal $y(t)$. The inverse transform is given by:

$$y(t) = \int_{-\infty}^{+\infty} Y(f)\exp(j2\pi ft)df \tag{A.17}$$

or, $y(t) = \mathcal{F}^{-1}Y(f)$

Note that according to the definition given by Equation A.13, the Fourier spectrum $Y(f)$ is defined for the entire frequency range $f(-\infty,+\infty)$ which includes negative values. This is termed the *two-sided spectrum*. Since, in practical applications it is not possible to have "negative frequencies," the *one-sided spectrum* is usually defined only for the frequency range $f(0, \infty)$.

In order that a two-sided spectrum have the same amount of *power* as a one-sided spectrum, it is necessary to make the one-sided spectrum double the two-sided spectrum for $f>0$.

If the signal is not sufficiently transient (fast-decaying or damped), the infinite integral given by Equation A.13 might not exist, but the corresponding Laplace transform might still exist.

A.4.1 Frequency-Response Function (Frequency Transfer Function)

The Fourier integral transform of the impulse-response function is given by

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt \tag{A.18}$$

where f is the *cyclic frequency* (measured in cycles/s or Hertz). This is known as the frequency-response function (or, frequency transfer function) of a system. Fourier transform operation is denoted as $\mathcal{F}h(t)=H(f)$. In view of the fact that $h(t)=0$ for $t<0$, the lower limit of integration in Equation A.18 could be made zero. Then, from Equation A.9, it is clear that $H(f)$ is obtained simply by setting $s=j2\pi f$ in $H(s)$. Hence, strictly speaking, we should use the notation $H(j2\pi f)$ and not $H(f)$. But for the notational simplicity we denote $H(j2\pi f)$ by $H(f)$. Furthermore, since the angular frequency $\omega=2\pi f$, we can express the frequency response function by $H(j\omega)$, or simply by $H(\omega)$ for the notational convenience. It should be noted that the frequency-response function, like the (Laplace) transfer function, is a complete representation of a linear, constant-parameter system. In view of the fact that both $u(t)=0$ and $y(t)=0$ for $t<0$, we can write the Fourier transforms of the input and the output of a system directly by setting $s=j2\pi f=j\omega$ in the corresponding Laplace transforms.

Then, from Equation A.8, we have

$$Y(f)=H(f)U(f) \tag{A.19}$$

Note: Sometimes for notational convenience, the same lowercase letters are used to represent the Laplace and Fourier transforms as well as the original time-domain variables.

If the Fourier integral transform of a function exists, then its Laplace transform also exists. The converse is not generally true, however, because of poor convergence of the Fourier integral in comparison to the Laplace integral. This arises from the fact that the factor $\exp(-\sigma t)$ is not present in the Fourier integral. For a physically realizable, linear, constant-parameter system, $H(f)$ exists even if $U(f)$ and $Y(f)$ do not exist for a particular input. The experimental determination of $H(f)$, however, requires system stability. For the nth-order system given by Equation A.10, the frequency-response function is determined by setting $s=j2\pi f$ in Equation A.12 as

$$H(f) = \frac{b_0 + b_1 j2\pi f + \cdots + b_m (j2\pi f)^m}{a_0 + a_1 j2\pi f + \cdots + a_n (j2\pi f)^n} \tag{A.20}$$

This, generally, is a complex function of f, which has a magnitude denoted by $|H(f)|$ and a phase angle denoted by $\angle H(f)$.

A.5 The *s*-Plane

We have noted that the Laplace variable s is a complex variable, with a real part and an imaginary part. Hence, to represent it we will need two axes at right angles to each other—the real axis and the imaginary axis. These two axes from a plane, which is called the s-plane. Any general value of s (or, any variation or trace of s) may be marked on the s-plane.

A.5.1 An Interpretation of Laplace and Fourier Transforms

In the Laplace transformation of a function $f(t)$ we multiply the function by e^{-st} and integrate with respect to t. This process may be interpreted as determining the "components" $F(s)$ of $f(t)$ in the "direction" e^{-st} where s is a complex variable. All such components $F(s)$ should be equivalent to the original function $f(t)$.

In the Fourier transformation of $f(t)$ we multiply it by $e^{-j\omega t}$ and integrate with respect to t. This is the same as setting $s=j\omega$. Hence, the Fourier transform of $f(t)$ is $F(j\omega)$. Furthermore, $F(j\omega)$ represents the components of $f(t)$ that are in the direction of $e^{-j\omega t}$. Since $e^{-j\omega t}=\cos\omega t - j\sin\omega t$, in the Fourier transformation what we do is to determine the sinusoidal components of frequency ω, of a time function $f(t)$. Since s is complex $F(s)$ is also complex and so is $F(j\omega)$. Hence they all will have a real part and an imaginary part.

A.5.2 Application in Circuit Analysis

The fact that $\sin\omega t$ and $\cos\omega t$ are 90° out of phase is further confirmed in view of

$$e^{j\omega t}=\cos\omega t+j\sin\omega t \tag{A.21}$$

Consider the *RLC* circuit shown in Figure A.2. For the capacitor, the current (i) and the voltage (v) are related through

$$i = C\frac{dv}{dt} \tag{A.22}$$

If the voltage $v=v_0\sin\omega t$, the current $i=v_0\omega C\cos\omega t$. Note that the magnitude of v/i is $1/\omega C$ (or, $1/2\pi fC$ where $\omega=2\pi f$; f is the cyclic frequency and ω is the angular frequency). But v and i are out of phase by 90°. In fact, in the case of a capacitor, i leads v by 90°. The equivalent circuit resistance of a capacitance is called *reactance*, and is given by

$$X_C = \frac{1}{2\pi fC} \tag{A.23}$$

$$= \frac{1}{\omega C} \tag{A.24}$$

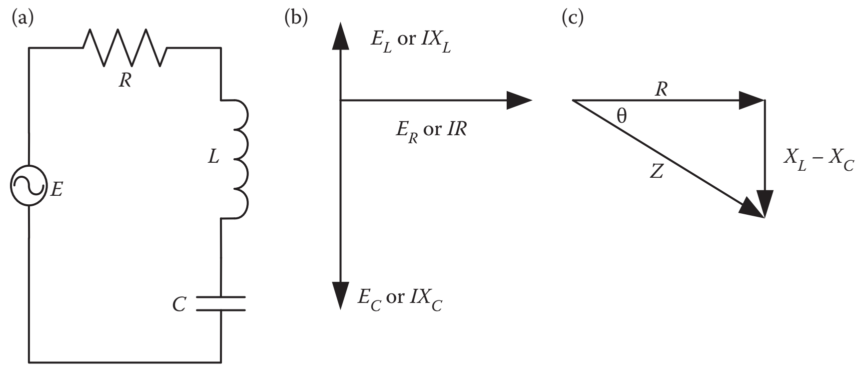


FIGURE A.2
(a) Series *RLC* circuit. (b) Phases of voltage drops. (c) Impedance triangle.

Note that this parameter changes with the frequency.

We cannot add the reactance of the capacitor and the resistance of the resistor algebraically; we must add them vectorialy because the voltages across a capacitor and resistor in series are not in phase, unlike in the case of a resistor. Also, the resistance in a resistor does not change with frequency. In a series circuit, as in Figure A.2, the current is identical in each element, but the voltages differ in both amplitude and phase; in a parallel circuit, the voltages are identical, but the currents differ in amplitude and phase.

Similarly, for an inductor

$$v = L\frac{di}{dt} \tag{A.25}$$

The corresponding reactance is

$$X_L = \omega L = 2\pi f L \tag{A.26}$$

If the voltage (E) across R in Figure A.2a is in the direction shown in Figure A.2b (i.e., pointing to the right), then the voltage across the inductor L must point upwards (90° leading) and the voltage across the capacitor C must point down (90° lagging). Since the current (I) is identical in each component of a series circuit, we see the directions of IR, IX_L and IX_C as in Figure A.2b, giving the impedance triangle shown in Figure A.2c.

To express these reactances in the s domain, we simply substitute s for $j\omega$:

$$-jX_C = \frac{1}{sC}$$

$$jX_L = s_L$$

The series impedance of the *RLC* circuit can be expressed as

$$Z = R + jX_L - jX_C = R + sL + \frac{1}{sC}$$

In this discussion, note the use of $\sqrt{-1}$ or j, to indicate a 90° phase change.