Appendix C
Review of Linear Algebra

Linear algebra, the algebra of *sets*, *vectors*, and *matrices*, is useful in the study of mechanical vibration and control systems in general and the state-space approach in particular. In practical mechanical vibration systems, interactions among various components are inevitable. There are many response variables associated with many excitations. Then, it is convenient to consider all excitations (inputs) simultaneously as a single variable, and also all responses (outputs) as a single variable. Use of *linear algebra* makes the analysis of such a system convenient. The subject of linear algebra is complex and is based on a rigorous mathematical foundation. In this appendix we will review the basics of vectors and matrices.

C.1 VECTORS AND MATRICES

In the analysis of mechanical vibration systems, vectors and matrices will be useful in both time and frequency domains. First, consider the time domain formulation of a mechanical system. For a single-degree-of-freedom (single-DoF)system with a single forcing excitation $f(t)$ and a corresponding single displacement response y, the dynamic equation would be

$$m\ddot{y} + c\dot{y} + ky = f(t) \tag{C.1}$$

Note that, in this single-DoF case, the quantities f, y, m, c, and k are *scalars*. If the system has n degrees of freedom, with excitation forces $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ and associated displacement responses y_1, y_2, ..., y_n, the equations of motion may be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{y}} + \boldsymbol{C}\dot{\boldsymbol{y}} + \boldsymbol{K}\boldsymbol{y} = \boldsymbol{f}(t) \tag{C.2}$$

in which

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \text{ displacement vector (}n\text{th order column vector)}$$

$$\boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \text{ forcing excitation vector (}n\text{th order column vector)}$$

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & & & \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} = \text{ mass matrix } (n \times n \text{ square matrix})$$

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \text{ damping matrix } (n \times n \text{ square matrix})$$

$$K = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{bmatrix} = \text{ stiffness matrix } (n \times n \text{ square matrix})$$

In this manner, vectors and matrices are introduced into the formulation of a multi-degree-of-freedom mechanical system. Further vector-matrix concepts will enter into the picture in subsequent analysis of the system; for example, in modal analysis.

Next consider the frequency-domain formulation. In the single-input-single-output (SISO) case, the system equation may be given as

$$y = Gu \tag{C.3}$$

in which

u = frequency spectrum (Fourier spectrum) of the forcing excitation (input)

y = frequency spectrum (Fourier spectrum) of the response (output)

G = frequency transfer function (frequency response function) of the system.

The quantities u, y and G are *scalars* because each one is a single quantity, and not a collection of several quantities.

Next, consider a multi-input-multi-output (MIMO) system having two excitations u_1 and u_2, and two responses y_1 and y_2; each y_i now depends on both u_1 and u_2. It follows that we need four transfer functions to represent all the excitation-response relationships that exist in this system. We use the four transfer functions (G_{11}, G_{12}, G_{21} and G_{22}). For example, the transfer function G_{12} relates the excitation u_2 to the response y_1. The associated two equations that govern the system are:

$$\begin{aligned} y_1 &= G_{11}u_1 + G_{12}u_2 \\ y_2 &= G_{21}u_1 + G_{22}u_2 \end{aligned} \tag{C.4}$$

Instead of considering the two excitations (two inputs) as two separate quantities, we can consider them as a single ñvectorò $\boldsymbol{u}$ having the two components u_1 and u_2. As before, we can write this as a column consisting of the two elements:

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In this case we have a ñcolumn vector.ò Alternately, we can write a ñrow vectorò as

$$\boldsymbol{u} = [u_1, u_2]$$

But, the column-vector representation is more common.

Similarly, we can express the two outputs y_1 and y_2 as a vector $\boldsymbol{y}$. Consequently, we have the column vector

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

or the row vector

$$\boldsymbol{y} = [y_1, y_2]$$

It should be kept in mind that the order in which the components (or elements) are given is important since the vector $[u_1, u_2]$ is not equal to the vector $[u_2, u_1]$. In other words, a vector is an ñorderedò collection of quantities.

Summarizing, we can express a collection of quantities, in an orderly manner, as a single vector. Each quantity in the vector is known as a *component* or an *element* of the vector. What each component means will depend on the particular situation. For example, in a dynamic system it may represent a quantity such as voltage, current, force, velocity, pressure, flow rate, temperature, or heat transfer rate. The number of components (elements) in a vector is called the *order*, or *dimension* of the vector.

Next let us introduce the concept of a matrix using the frequency domain example given above. Note that we needed four transfer functions to relate the two excitations to the two responses. Instead of considering these four quantities separately we can express them as a single matrix $\boldsymbol{G}$ having four elements. Specifically, the *transfer function matrix* for the present example is

$$\boldsymbol{G} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Note that our matrix has two rows and two columns. Hence the size or order of the matrix is 2×2. Since the number of rows is equal to the number of columns in this example, we have a *square matrix*. If the number of rows is not equal to the number of columns, we have a *rectangular matrix*. Actually, we can interpret a matrix as a collection of vectors. Hence, in the previous example, the matrix $\boldsymbol{G}$ is an assembly of the two column vectors

$$\begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix} \text{and} \begin{bmatrix} G_{12} \\ G_{22} \end{bmatrix}$$

or, alternatively, an assembly of the two row vectors

$$[G_{11}, G_{12}]$$

and

$$[G_{21}, G_{22}]$$

C.2 VECTOR-MATRIX ALGEBRA

The advantage of representing the excitations and the responses of a mechanical vibration system as the vectors $\boldsymbol{u}$ and $\boldsymbol{y}$, and the transfer functions as the matrix $\boldsymbol{G}$ is clear from the fact that the excitation-response (input-output) equations can be expressed as the single equation

$$\boldsymbol{y} = \boldsymbol{G}\boldsymbol{u} \tag{C.5}$$

instead of the collection of scalar equations (C.4).

Hence the response vector $\boldsymbol{y}$ is obtained by ñpre-multiplyingò the excitation vector $\boldsymbol{u}$ by the transfer function matrix $\boldsymbol{G}$. Of course, certain rules of vector-matrix multiplication have to be agreed upon in order that this single equation is consistent with the two scalar equations given by (C.4). Also, we have to agree upon rules for the addition of vectors or matrices.

A vector is a special case of a matrix. Specifically, a 3rd-order column vector is a matrix having 3 rows and one column. Hence it is a 3x1 matrix. Similarly, a 3rd-order row vector is a matrix having one row and three columns. Accordingly, it is a 1x3 matrix. It follows that we only need to know matrix algebra, and the vector algebra will follow from the results for matrices.

C.2.1 Matrix Addition and Subtraction

Only matrices of the same size can be added. The result (sum) will also be a matrix of the same size. In matrix addition, we add the corresponding elements (i.e., the elements at the same position) in the two matrices, and write the results at the corresponding places in the resulting matrix.

As an example, consider the 2×3 matrix

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 6 & -2 \end{bmatrix}$$

and a second matrix

$$\boldsymbol{B} = \begin{bmatrix} 2 & 1 & -5 \\ 0 & -3 & 2 \end{bmatrix}$$

The sum of these two matrices is given by

$$A + B = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 0 \end{bmatrix}$$

The order in which the addition is done is immaterial. Hence

$$A + B = B + A \tag{C.6}$$

In other words, matrix addition is *commutative*.

Matrix subtraction is defined just like matrix addition, except the corresponding elements are subtracted. An example is given below:

$$\begin{bmatrix} -1 & 2 \\ 3 & 0 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 2 & -1 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$$

C.2.2 Null Matrix

The null matrix is a matrix whose elements are all zeros. Hence when we add a null matrix to an arbitrary matrix the result is equal to the original matrix. We can define a *null vector* in a similar manner. We can write

$$\boldsymbol{A} + \boldsymbol{0} = \boldsymbol{A} \tag{C.7}$$

As an example, the 2×2 null matrix is:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

C.2.3 Matrix Multiplication

Consider the product ***AB*** of the two matrices ***A*** and ***B***. Let us write this as:

$$\boldsymbol{C} = \boldsymbol{AB} \tag{C.8}$$

We say that ***B*** is *pre-multiplied* by ***A*** or, equivalently, ***A*** is *post-multiplied* by ***B***. For this multiplication to be possible, the number of columns in ***A*** has to be equal to the number of rows in ***B***. Then, the number of rows of the product matrix ***C*** is equal to the number of rows in ***A***, and the number of columns in ***C*** is equal to the number of columns in ***B***.

The actual multiplication is done by multiplying the elements in a given row (say the *i*th row) of ***A*** by the corresponding elements in a given column (say the, *j*th column) of ***B*** and summing these products. The result is the element c_{ij} of the product matrix ***C***. Note that c_{ij} denotes the element that is common to the *i*th row and the *j*th column of matrix ***C***. So, we have

$$\boldsymbol{c}_{ij} = \sum_k \boldsymbol{a}_{ik} \boldsymbol{b}_{kj} \tag{C.9}$$

As an example, suppose:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 4 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & -1 & 2 & 4 \\ 2 & 3 & -4 & 2 \\ 5 & -3 & 1 & 0 \end{bmatrix}$$

Note that the number of columns in $\boldsymbol{A}$ is equal to 3 and the number of rows in $\boldsymbol{B}$ is also equal to 3. Hence we can perform the pre-multiplication of $\boldsymbol{B}$ by $\boldsymbol{A}$. For example

$$c_{11} = 1 \times 1 + 2 \times 2 + (-1) \times 5 = 0$$
$$c_{12} = 1 \times (-1) + 2 \times 3 + (-1) \times (-3) = 8$$
$$c_{13} = 1 \times 2 + 2 \times (-4) + (-1) \times 1 = -7$$
$$c_{14} = 1 \times 4 + 2 \times 2 + (-1) \times 0 = 8$$
$$c_{21} = 3 \times 1 + (-3) \times 2 + 4 \times 5 = 17$$
$$c_{22} = 3 \times (-1) + (-3) \times 3 + 4 \times (-3) = -24$$

and so on. The product matrix is

$$\boldsymbol{C} = \begin{bmatrix} 0 & 8 & -7 & 8 \\ 17 & -24 & 22 & 6 \end{bmatrix}$$

It should be noted that both products $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are not always defined, and even when they are defined, the two results are not equal in general. Unless both $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same order, the two product matrices will not be of the same order.

Summarizing, matrix multiplication is not commutative:

$$\boldsymbol{AB} \neq \boldsymbol{BA} \tag{C.10}$$

C.2.4 Identity Matrix

An identity matrix (or unity matrix) is a square matrix whose diagonal elements are all equal to 1 and all the remaining elements are zeros. This matrix is denoted by $\boldsymbol{I}$. For example, the 3rd-order identity matrix is

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easy to see that when any matrix is multiplied by an identity matrix (provided, of course, that the multiplication is possible) the product is equal to the original matrix. Thus

$$\boldsymbol{A\,I} = \boldsymbol{I\,A} = \boldsymbol{A} \tag{C.11}$$

C.3 MATRIX INVERSE

An operation similar to scalar division can be defined in terms of the inverse of a matrix. A proper inverse is defined only for a square matrix and, even for a square matrix, an inverse might not exist. The inverse of a matrix is defined as follows:

Suppose that a square matrix $\boldsymbol{A}$ has the inverse $\boldsymbol{B}$. Then these must satisfy the equation:

$$\boldsymbol{AB} = \boldsymbol{I} \qquad \text{(C.12)}$$

or, equivalently

$$\boldsymbol{BA} = \boldsymbol{I} \qquad \text{(C.13)}$$

where $\boldsymbol{I}$ is the identity matrix, as defined before.

The inverse of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$. The inverse exists for a matrix if and only if the *determinant* of the matrix is nonzero. Such matrices are termed *nonsingular*. We shall discuss the determinant in a later subsection of this Appendix. But, before explaining a method for determining the inverse of a matrix let us verify that

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

To show this we simply multiply the two matrices and show that the product is the second-order unity matrix. Specifically,

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

C.3.1 Matrix Transpose

The transpose of a matrix is obtained by simply interchanging the rows and the columns of the matrix. The transpose of $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{\boldsymbol{T}}$.

For example, the transpose of the 2×3 matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 2 & 0 \end{bmatrix}$$

is the 3×2 matrix

$$\boldsymbol{A}^{\boldsymbol{T}} = \begin{bmatrix} 1 & -2 \\ -2 & 2 \\ 3 & 0 \end{bmatrix}$$

Note that the first row of the original matrix has become the first column of the transposed matrix, and the second row of the original matrix has become the second column of the transposed matrix.

If $\boldsymbol{A}^{\boldsymbol{T}} = \boldsymbol{A}$ then we say that the matrix $\boldsymbol{A}$ is *symmetric*. Another useful result on the matrix transpose is expressed by

$$(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}} \tag{C.14}$$

If follows that the transpose of a matrix product is equal to the product of the transposed matrices, taken in the reverse order.

C.3.2 Trace of a Matrix

The trace of a square matrix is given by the sum of the diagonal elements. The trace of matrix $\boldsymbol{A}$ is denoted by tr($\boldsymbol{A}$).

$$\mathrm{tr}(\boldsymbol{A}) = \sum_i \boldsymbol{a}_{i\,i} \tag{C.15}$$

For example, the trace of the matrix

$$\boldsymbol{A} = \begin{bmatrix} -2 & 3 & 0 \\ 4 & -4 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

is given by

$$\mathrm{tr}(\boldsymbol{A}) = (-2) + (-4) + 3 = -3$$

C.3.3 Determinant of a Matrix

The determinant is defined only for a square matrix. It is a scalar value computed from the elements of the matrix. The determinant of a matrix $\boldsymbol{A}$ is denoted by det($\boldsymbol{A}$) or $|\boldsymbol{A}|$.

Instead of giving a complex mathematical formula for the determinant of a general matrix in terms of the elements of the matrix, we now explain a way to compute the determinant.

First consider the 2×2 matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Its determinant is given by

$$\det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}$$

Next consider the 3×3 matrix

$$
\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
$$

Its determinant can be expressed as

$$
\det(\boldsymbol{A}) = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}
$$

where, the *minors* of the associated matrix elements are defined as

$$
M_{11} = \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad M_{12} = \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
$$

$$
M_{13} = \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
$$

Note that M_{ij} is the determinant of the matrix obtained by deleting the *i*th row and the *j*th column of the original matrix. The quantity M_{ij} is known as the *minor* of the element a_{ij} of the matrix $\boldsymbol{A}$. If we attach a proper sign to the minor depending on the position of the corresponding matrix element, we have a quantity known as the *cofactor*. Specifically, the cofactor C_{ij} corresponding to the minor M_{ij} is given by

$$
C_{ij} = (-1)^{i+j} M_{ij} \tag{C.16}
$$

Hence the determinant of the 3×3 matrix may be given by

$$
\det(\boldsymbol{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}
$$

Note that in the two formulas given above for computing the determinant of a 3×3 matrix, we have expanded along the first row of the matrix. We get the same answer, however, if we expand along any row or any column. Specifically, when expanded along the *i*th row we have

$$
\det(\boldsymbol{A}) = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}
$$

Similarly, if we expand along the *j*th column we have

$$
\det(\boldsymbol{A}) = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}
$$

These ideas of computing a determinant can be easily extended to 4×4 and higher-order matrices in a straightforward manner. Hence, we can write

$$
\det(\boldsymbol{A}) = \sum_j a_{ij}C_{ij} = \sum_i a_{ij}C_{ij} \tag{C.17}
$$

C.3.4 Adjoint of a Matrix

The adjoint of a matrix is the transpose of the matrix whose elements are the cofactors of the corresponding elements of the original matrix. The adjoint of matrix ***A*** is denoted by adj(***A***).

As an example, in the 3×3 case we have

$$\operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

In particular, it is easily seen that the adjoint of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

is given by

$$\operatorname{adj}(\,\boldsymbol{A}\,) = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 2 & 1 \\ 7 & -2 & 3 \end{bmatrix}^T$$

Accordingly we have

$$\operatorname{adj}(\boldsymbol{A}) = \begin{bmatrix} 1 & -3 & 7 \\ 2 & 2 & -2 \\ -3 & 1 & 3 \end{bmatrix}$$

Hence, in general

$$\operatorname{adj}(\boldsymbol{A}) = \left[C_{ij}\right]^T \tag{C.18}$$

C.3.5 Inverse of a Matrix

At this juncture it is appropriate to give a formula for the inverse of a square matrix. Specifically

$$\boldsymbol{A}^{-1} = \frac{\operatorname{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})} \tag{C.19}$$

Hence in the 3×3 matrix example given before, since we have already determined the adjoint, it remains only to compute the determinant in order to obtain the inverse. Now expanding along the first row of the matrix, the determinant is given by

$$\det(\boldsymbol{A}) = 1 \times 1 + 2 \times 2 + (-1) \times (-3) = 8$$

Accordingly, the inverse is given by

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 1 & -3 & 7 \\ 2 & 2 & -2 \\ -3 & 1 & 3 \end{bmatrix}$$

For two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ we have

$$\left(\boldsymbol{AB}\right)^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \tag{C.20}$$

As a final note, if the determinant of a matrix is zero, the matrix does not have an inverse. Then we say that the matrix is *singular*. Some important matrix properties are summarized in Box C.1.

Box C.1 Summary of Matrix Properties.

Addition: $\boldsymbol{A}_{m\times n} + \boldsymbol{B}_{m\times n} = \boldsymbol{C}_{m\times n}$

Multiplication: $\boldsymbol{A}_{m\times n}\boldsymbol{B}_{n\times r} = \boldsymbol{C}_{m\times r}$

Identity: $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A} \Rightarrow \boldsymbol{I}$ is the identity matrix

<u>Note</u>: $\boldsymbol{AB} = \boldsymbol{0} \not\Rightarrow \boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$ in general

Transposition: $\boldsymbol{C}^T = \left(\boldsymbol{AB}\right)^T = \boldsymbol{B}^T\boldsymbol{A}^T$

Inverse: $\boldsymbol{AP} = \boldsymbol{I} = \boldsymbol{PA} \Rightarrow \boldsymbol{A} = \boldsymbol{P}^{-1}$and $\boldsymbol{P} = \boldsymbol{A}^{-1}$

$\left(\boldsymbol{AB}\right)^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Commutativity: $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general

Associativity: $\left(\boldsymbol{AB}\right)\boldsymbol{C} = \boldsymbol{A}\left(\boldsymbol{BC}\right)$

Distributivity: $\boldsymbol{C}\left(\boldsymbol{A}+\boldsymbol{B}\right) = \boldsymbol{CA}+\boldsymbol{CB}$

Distributivity: $\left(\boldsymbol{A}+\boldsymbol{B}\right)\boldsymbol{D} = \boldsymbol{AD}+\boldsymbol{BD}$

C.4 VECTOR SPACES

C.4.1 Field (*F*)

Consider a set of scalars.

If for any α and β from the set, $\alpha+\beta$ and $\alpha\beta$ are also elements in the set and if:

1. $\alpha+\beta = \beta+\alpha$ and $\alpha\beta = \beta\alpha$ (Commutativity)
2. $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (Associativity)
3. $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$ (Distributivity)

are satisfied,

and if:

1. Identity elements 0 and 1 exist in the set such that $\alpha + 0 = \alpha$ and $1\alpha = \alpha$
2. Inverse elements exist in the set such that $\alpha + (-\alpha) = 0$ and $\alpha \cdot \alpha^{-1} = 1$

then, the set is a field.

E.g.: The set of real numbers.

C.4.2 Vector Space (*L*)

Properties:

1. Vector addition ($\boldsymbol{x} + \boldsymbol{y}$) and scalar multiplication ($\alpha\boldsymbol{x}$) are defined.
2. Commutativity: $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{y}+\boldsymbol{x}$
   Associativity: $(\boldsymbol{x}+\boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y}+\boldsymbol{z})$
   are satisfied.
3. Unique null vector $\mathbf{0}$ and negation ($-\boldsymbol{x}$) exist such that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$
   $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}.$
4. Scalar multiplication satisfies

$$\alpha(\beta\boldsymbol{x}) = (\alpha\beta)\boldsymbol{x} \qquad \text{(Associativity)}$$

$$\left.\begin{aligned}\alpha(\boldsymbol{x}+\boldsymbol{y}) &= \alpha\boldsymbol{x} + \beta\boldsymbol{y}\\ (\alpha+\beta)\boldsymbol{x} &= \alpha\boldsymbol{x} + \beta\boldsymbol{x}\end{aligned}\right\} \qquad \text{(Distribuitivity)}$$

$$1\boldsymbol{x} = \boldsymbol{x}, \quad 0\boldsymbol{x} = \mathbf{0}$$

Special Case: Vector space L^n has vectors with *n* elements from the field *F*.

Consider $$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$$

Then

$$\boldsymbol{x}+\boldsymbol{y} = \begin{bmatrix} x_1+y_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n+y_n \end{bmatrix} = \boldsymbol{y}+\boldsymbol{x} \quad \text{and} \quad \alpha\boldsymbol{x} = \begin{bmatrix} \alpha x_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha x_n \end{bmatrix}$$

C.4.3 Subspace *S* of *L*

1. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in S then $\boldsymbol{x} + \boldsymbol{y}$ is also in S.
2. If $\boldsymbol{x}$ is in S and α is in F then $\alpha\boldsymbol{x}$ is also in S.

C.4.4 Linear Dependence

Consider the set of vectors: $x_1, x_2, \ldots, x_n$

They are linearly independent if any one of these vectors cannot be expressed as a linear combination of one or more remaining vectors.

Necessary and sufficient condition for linear independence:

$$\alpha_1 \boldsymbol{x}_1 + \alpha_2 \boldsymbol{x}_2 + \ldots \alpha_n \boldsymbol{x}_n = \mathbf{0} \qquad \text{(C.21)}$$

gives $\alpha = \mathbf{0}$ (trivial solution) as the only solution.

E.g.: $$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}; \quad \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}; \quad \boldsymbol{x}_3 = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

These vectors are not linearly independent because, $\boldsymbol{x}_1 + 2\boldsymbol{x}_2 = \boldsymbol{x}_3$.

C.4.5 Bases and Dimension of a Vector Space

1. If a set of vectors can be combined to form any vector in L then that set of vectors is said to *span* the vector space L (i.e., a generating system of vectors).

2. If the spanning vectors are all linearly independent, then this set of vectors is a *basis* for that vector space.

3. The number of vectors in the basis = dimension of the vector space.

Note: Dimension of a vector space is not necessarily the order of the vectors.

E.g.: Consider two intersecting 3rd order vectors. The will form a basis for the plane (two dimensional) that contains the two vectors. Hence, the dimension of the vector space = 2, but the order of each vector in the basis = 3.

Note: L^n is spanned by n linearly independent vectors

$$\Rightarrow \dim(L^n) = n$$

E.g.: $\begin{bmatrix}1\\0\\0\\\cdot\\\cdot\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\cdot\\\cdot\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\\cdot\\\cdot\\0\\1\end{bmatrix}$

C.4.6 Inner Product

$$(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^H \boldsymbol{x} \tag{C.22}$$

where ñHò denotes the *hermitian transpose* (i.e., complex conjugate and transpose). Hence $\boldsymbol{y}^H = (\boldsymbol{y}^*)^T$ where ( )* denotes complex conjugation.

Note:

1. $(\boldsymbol{x},\boldsymbol{x}) \geq 0$ and $(\boldsymbol{x},\boldsymbol{x}) = 0$ if and only if (iff) $\boldsymbol{x} = \boldsymbol{0}$

2. $(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{y},\boldsymbol{x})^*$

3. $(\lambda\boldsymbol{x}, \boldsymbol{y}) = \lambda(\boldsymbol{x}, \boldsymbol{y})$
   $(\boldsymbol{x}, \lambda\boldsymbol{y}) = \lambda^*(\boldsymbol{x}, \boldsymbol{y})$

4. $(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x}, \boldsymbol{y}) + (\boldsymbol{x}, \boldsymbol{z})$

C.4.7 Norm

Properties:

$\|\boldsymbol{x}\| \geq 0$ and $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x}=\boldsymbol{0}$

$\|\lambda\boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$ for any scalar λ

$\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

E.g.: Euclidean norm: $$\|\boldsymbol{x}\| = \boldsymbol{x}^H \boldsymbol{x} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \tag{C.23}$$

Unit Vector: $\|\boldsymbol{x}\| = 1$

Normalization: $\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \hat{\boldsymbol{x}}$

Angle Between Vectors: We have $\cos\theta = \frac{(\boldsymbol{x},\boldsymbol{y})}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} = (\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})$ (C.24)

Where θ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$.

Orthogonal Vectors: iff $(\boldsymbol{x},\boldsymbol{y}) = 0$ (C.25)

Note: n orthogonal vectors in L^n are linearly independent and span L^n, and form a basis for L^n

C.4.8 Gram-Schmidt Orthogonalization

Given a set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ that are linearly independent in L^n, we construct a set of orthonormal (orthogonal and normalized) vectors $\hat{\boldsymbol{y}}_1, \hat{\boldsymbol{y}}_2, \cdots \hat{\boldsymbol{y}}_n$ which are linear combinations of $\hat{\boldsymbol{x}}_i$

Start $\hat{\boldsymbol{y}}_1 = \hat{\boldsymbol{x}}_1 = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}$

Then $\boldsymbol{y}_i = \boldsymbol{x}_i - \sum_{j=1}^{i-1} (\boldsymbol{x}_i, \hat{\boldsymbol{y}}_j)\hat{\boldsymbol{y}}_j$ for $i = 1,2,\ldots,n$

Normalize $\boldsymbol{y}_i$ to produce $\hat{\boldsymbol{y}}_i$

C.4.9 Modified Gram-Schmidt Procedure

In each step compute new vectors that are orthogonal to the just-computed vector.

Step 1: $\hat{\boldsymbol{y}}_1 = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|}$ as before.

Then $\boldsymbol{x}_i^{(1)} = \boldsymbol{x}_i - (\hat{\boldsymbol{y}}_1, \boldsymbol{x}_i)\hat{\boldsymbol{y}}_1$ for $i = 2,3,\ldots,n$

$\hat{\boldsymbol{y}}_i = \frac{\boldsymbol{x}_i^{(1)}}{\left\|\boldsymbol{x}_i^{(1)}\right\|}$ for $i = 2, 3, \ldots n$

and $\boldsymbol{x}_i^{(2)} = \boldsymbol{x}_i^{(1)} - (\hat{\boldsymbol{y}}_2, \boldsymbol{x}_i^{(1)})\hat{\boldsymbol{y}}_2$, $i = 3,4,\ldots, n$ and *so* on.

C.5 DETERMINANTS

Now, let us address several analytical issues of the determinant of a square matrix. Consider the matrix,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \cdot & \cdot & a_{1n} \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & \cdot & \cdot & a_{nn} \end{bmatrix}$$

Minor of $a_{ij} = M_{ij} =$ determinant of matrix formed by deleting the ith row and the j^{th} column of the original matrix.

Cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$

cof ($\boldsymbol{A}$) = cofactor matrix of $\boldsymbol{A}$

adj ($\boldsymbol{A}$) = adjoint $\boldsymbol{A}$ = (cof $\boldsymbol{A}$)^T

C.5.1 Properties of Determinant of a Matrix

1. Interchange two rows (columns) $\Rightarrow$ Determinant sign changes.

2. Multiply one row (column) by $\alpha \Rightarrow \alpha$ det ().

3. Add a [$\alpha \times$ row (column)] to a second row (column) $\Rightarrow$ determinant unchanged.

4. Identical rows (columns) $\Rightarrow$ zero determinant.

5. For two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, det($\boldsymbol{AB}$) = det($\boldsymbol{A}$) det($\boldsymbol{B}$).

C.5.2 Rank of a Matrix

Rank $\boldsymbol{A}$ = number of linearly independent columns = number of linearly independent rows
= dim (column space) = dim (row space)

Here ñdimò denotes the ñdimension of.ò

C.6 SYSTEM OF LINEAR EQUATIONS

Consider the set of linear algebraic equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m
\end{aligned}
$$

We need to solve for $x_1, x_2, \cdots, x_n$.

This problem can be expressed in the vector-matrix form:

$$\boldsymbol{A}_{m\times n}\boldsymbol{x}_n = \boldsymbol{c}_m \qquad \boldsymbol{B} = [\boldsymbol{A}, \boldsymbol{c}]$$

Solution exists iff rank [$\boldsymbol{A}$, $\boldsymbol{c}$] = rank [$\boldsymbol{A}$]

Two cases can be considered:

Case 1: If $m \geq n$ and rank [$\boldsymbol{A}$] = n $\Rightarrow$ unique solution for $\boldsymbol{x}$.

Case 2: If $m \leq n$ and rank [$\boldsymbol{A}$] = m $\Rightarrow$ infinite number of solutions for $\boldsymbol{x}$

$$\boldsymbol{x} = \boldsymbol{A}^H(\boldsymbol{A}\boldsymbol{A}^H)^{-1}\boldsymbol{C} \Leftarrow \text{minimum norm form}$$

Specifically, out of the infinite possibilities, this is the solution that minimizes the norm $\boldsymbol{x}^H\boldsymbol{x}$.

Note that the superscript ñHò denotes the ñhermitian transpose,ò which is the transpose of the complex conjugate of the matrix

E.g.:
$$\boldsymbol{A} = \begin{bmatrix} 1+j & 2+3j & 6 \\ 3-j & 5 & -1-2j \end{bmatrix}$$

Then
$$\boldsymbol{A}^H = \begin{bmatrix} 1-j & 3+j \\ 2-3j & 5 \\ 6 & -1+2j \end{bmatrix}$$

If the matrix is real, its hermitian transpose is simply the ordinary transpose.

In general if rank $[\boldsymbol{A}] \leq n \Rightarrow$ infinite number of solutions.

The space formed by solutions $\boldsymbol{Ax} = \boldsymbol{0} \Rightarrow$ is called the *null space* dim (null space) = n-k where rank [$\boldsymbol{A}$]=k

C.7 QUADRATIC FORMS

Consider a vecor $\boldsymbol{x}$ and a square matrix $\boldsymbol{A}$. Then the function $Q(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{Ax})$ is called a quadratic form. For a real vector $\boldsymbol{x}$ and a real and symmetric matrix $\boldsymbol{A}$,

$$Q(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Ax}$$

Positive Definite Matrix: If $(\boldsymbol{x}, \boldsymbol{Ax}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{A}$ is said to be a positive definite matrix. Also, the corresponding quadratic form is also said to be positive definite.

Positive Semidefinite Matrix: If $(\boldsymbol{x}, \boldsymbol{Ax}) \geq 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{A}$ is said to be a positive semidefinite matrix. Note that in this case the quadratic form can assume a zero value for a non-zero $\boldsymbol{x}$. Also, the corresponding quadratic form is also said to be positive semidefinite.

Negative Definite Matrix: If $(\boldsymbol{x}, \boldsymbol{Ax}) < 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{A}$ is said to be a negative definite matrix. Also, the corresponding quadratic form is also said to be negative definite.

Negative Semidefinite Matrix: If $(\boldsymbol{x}, \boldsymbol{Ax}) \leq 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{A}$ is said to be a negative semidefinite matrix. Note that in this case the quadratic form can assume a zero value for a non-zero $\boldsymbol{x}$. Also, the corresponding quadratic form is also said to be negative semidefinite.

Note: If $\boldsymbol{A}$ is positive definite, then ï$\boldsymbol{A}$ is negative definite. If $\boldsymbol{A}$ is positive semidefinite, then ï$\boldsymbol{A}$ is negative semidefinite.

Principal Minors: Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Its principal minors are the determinants of the various matrices along the principal diagonal, as given by

$$\Delta_1 = a_{11}, \ \Delta_2 = \det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ \Delta_3 = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ and so on.}$$

Sylvester's Theorem: A matrix is positive if definite if all its principal minors are positive.

C.8 MATRIX EIGENVALUE PROBLEM

C.8.1 Characteristic Polynomial

Consider a square matrix $\boldsymbol{A}$. The polynomial

$$\Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}]$$

is called the characteristic polynomial of $\boldsymbol{A}$.

C.8.2 Characteristic Equation

The polynomial equation

$$\Delta(s) = \det[s\boldsymbol{I} - \boldsymbol{A}] = 0$$

is called the characteristic equation of the square matrix $\boldsymbol{A}$.

C.8.3 Eigenvalues

The roots of the characteristic equation of a square matrix $\boldsymbol{A}$ are the eigenvalues of $\boldsymbol{A}$. For an $n \times n$ matrix, there will be n eigenvalues.

C.8.4 Eigenvectors

The eigenvalue problem of a square matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

where, the objective is to solve for λ and the corresponding nontrivial (i.e., non-zero) solutions for $\boldsymbol{v}$. The problem can be expressed as

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = \boldsymbol{0}$$

Note: If $\boldsymbol{v}$ is a solution of this equation, then any multiple $a\boldsymbol{v}$ of it is also a solution. Hence, an eigenvector is arbitrary up to a multiplication factor.

For a nontrivial (i.e., non-zero) solution to be possible for $\boldsymbol{v}$, one must have

$$\det[\lambda \boldsymbol{I} - \boldsymbol{A}] = 0$$

Since this is the characteristic equation of $\boldsymbol{A}$, as defined above, it is clear that the roots of λ are the eigenvalues of $\boldsymbol{A}$. The corresponding solutions for $\boldsymbol{v}$ are the eigenvectors of $\boldsymbol{A}$. For an $n \times n$ matrix, there will be n eigenvalues and n corresponding eigenvectors.

C.9 MATRIX TRANSFORMATIONS

C.9.1 Similarity Transformation

Consider a square matrix $\boldsymbol{A}$ and a nonsingular (and square) matrix $\boldsymbol{T}$. Then, the matrix obtained according to

$$\boldsymbol{B} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$$

is the similarity transformation of $\boldsymbol{A}$ by $\boldsymbol{T}$. The transformed matrix $\boldsymbol{B}$ has the same eigenvalues as the original matrix $\boldsymbol{A}$. Also, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be similar.

C.9.2 Orthogonal Transformation

Consider a square matrix $\boldsymbol{A}$ and another square matrix $\boldsymbol{T}$. Then, the matrix obtained according to

$$\boldsymbol{B} = \boldsymbol{T}^{T}\boldsymbol{A}\boldsymbol{T}$$

is the orthogonal transformation of $\boldsymbol{A}$ by $\boldsymbol{T}$.

If $T^{1} = T^{T}$ then the matrix T is said to be an orthogonal matrix. In this case, the similarity transformation and the orthogonal transformation become identical.

C.10 MATRIX EXPONENTIAL

The matrix exponential is given by the infinite series

$$\exp(\boldsymbol{A}t) = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2 t^2 + \cdots \tag{C.26}$$

exactly like the scalar exponential

$$\exp(\lambda t) = 1 + \lambda t + \frac{1}{2!}\lambda^2 t^2 + \cdots \tag{C.27}$$

The matrix exponential maybe determined by reducing the infinite series given in equation (C.26) into a finite matrix polynomial of order n - 1 (where, $\boldsymbol{A}$ is $n\times n$) by using the Cayley-Hamilton theorem.

Cayley-Hamilton Theorem: This theorem states that a matrix satisfies its own characteristic equation. The characteristic polynomial of $\boldsymbol{A}$ can be expressed as

$$\begin{aligned}\Delta(\lambda) &= \det(\boldsymbol{A} - \lambda\boldsymbol{I}) \\ &= a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0\end{aligned} \tag{C.28}$$

in which det( ) denotes determinant. The notation

$$\Delta(\boldsymbol{A}) = a_n\,\boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \cdots + a_0\boldsymbol{I} \tag{C.29}$$

is used. Then, by the Cayley-Hamilton thorem, we have

$$\boldsymbol{0} = a_n\,\boldsymbol{A}^n + a_{n-1}\boldsymbol{A}^{n-1} + \cdots + a_0\boldsymbol{I} \tag{C.30}$$

C.10.1 Computation of Matrix Exponential

Using Cayley-Hamilton theorem, we can obtain a finite polynomial expansion for $\exp(\boldsymbol{A}t)$. First we express (C.26) and (C.27) as

$$\exp(\boldsymbol{A}t) = S(\boldsymbol{A})\cdot\Delta(\boldsymbol{A}) + \alpha_{n-1}\boldsymbol{A}^{n-1} + \alpha_{n-2}\boldsymbol{A}^{n-2} + \cdots + \alpha_0\boldsymbol{I} \tag{C.31}$$

$$\exp(\lambda t) = S(\lambda)\cdot\Delta(\lambda) + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_0 \tag{C.32}$$

in which $S(.)$ is an appropriate infinite series, which is the result of dividing the exponential (infinite) series by the characteristic polynomial $\Delta(.)$.

Next, since $\Delta(\boldsymbol{A}) = \boldsymbol{0}$ by the Cayley-Hamiltion theorem, equation (C.31) becomes

$$\exp(\boldsymbol{A}t) = \alpha_{\boldsymbol{n}-1}\boldsymbol{A}^{\boldsymbol{n}-1} + \alpha_{\boldsymbol{n}-2}\boldsymbol{A}^{\boldsymbol{n}-2} + \cdots + \alpha_0 \boldsymbol{I} \tag{C.33}$$

Now it is just a matter of determining the coefficients $\alpha_0, \alpha_1, \cdots, \alpha_{\boldsymbol{n}-1}$, which are functions of time. This is done as follows. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of ***A*,** however, then, by definition,

$$\Delta(\lambda_i) = \det(\boldsymbol{A} - \lambda_i \boldsymbol{I}) = 0 \quad \text{for} \quad i = 1, 2, \cdots, n \tag{C.34}$$

Thus, from equation (C.32), we obtain

$$\exp(\lambda_i t) = \alpha_{n-1}\lambda_i^{n-1} + \alpha_{n-2}\lambda_i^{n-2} + \cdots + \alpha_0 \quad \text{for} \quad i = 1, 2, \cdots, n \tag{C.35}$$

If the eigenvalues are all distinct, equation (C.35) represents a set of n independent algebraic equations from which the n unknowns $\alpha_0, \alpha_1, \cdots, \alpha_{\boldsymbol{n}-1}$ could be determined. If some eigenvalues are repeated, the derivatives of the corresponding equations (C.35) have to be used as well.